Fast Converging Chernoff Approximations

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Definition of a C_0-semigroup

Let $\mathcal{F}$ be a Banach space, and $\mathscr{L}(\mathcal{F})$ be the space of all linear bounded operators on $\mathcal{F}$. Consider mapping $V\colon [0;+\infty) \to \mathscr{L}(\mathcal{F})$, which for every fixed $t \geq 0$ is a linear bounded operator $V(t)\colon \mathcal{F} \to \mathcal{F}$.

The family $(V(t))_{t\geq 0} \subset \mathscr{L}(\mathcal{F})$ is called C_0-**semigroup** iff the following holds:

1. $V(0) = I$, i.e. $V(0)f = f$ for all $f \in \mathcal{F}$;
2. $V(t+s) = V(t) \circ V(s)$ for any $t \geq 0, s \geq 0$;
3. V is continuous in strong operator topology, i.e. for any $f \in \mathcal{F}$ a mapping $t \longmapsto V(t)f$ is continuous.

For the C_0-semigroup, there is an analogue of the derivative at zero. This object is called its generator and is defined as follows.

Definition of a C_0-semigroup generator

By the **generator** of a C_0-semigroup of linear bounded operators in $\mathcal{F}$ we mean a linear operator L: $Dom(L) \to \mathcal{F}$ given by the formula

$$Lf = \lim_{t \to +0} \frac{V(t)f - f}{t},$$

defined on its domain $Dom(L)$, that is a dense subspace of $\mathcal{F}$ such that there exist a given limit where the limit is understood in the strong sense, i.e. it is defined in terms of the norm in space $\mathcal{F}$. The generator generates a C_0-semigroup, and one can use the notation $V(t) = e^{tL}$.

C_0-semigroup and linear evolution equations

let Q be some set. In the Cauchy problem for an evolution partial differential equation

$$\begin{cases} u'_t(t,x) = Lu(t,x) & \text{for} \quad t > 0, x \in Q, \\ u(0,x) = u_0(x) & \text{for} \quad x \in Q. \end{cases}$$

we can assume $U(t) = u(t,\cdot) = [x \longmapsto u(t,x)]$ and get the Cauchy problem for an ordinary differential equation:

$$\begin{cases} \frac{d}{dt}U(t) = LU(t) & \text{for} \quad t > 0, \\ U(0) = u_0. \end{cases}$$

It is known that if $u(t,\cdot) \in \mathcal{F}$ and there exists a C_0-semigroup with generator L, that is, if there is an exponential form the operator tL, then both problems have a solution

$$U(t) = e^{tL}u_0, \quad u(t,x) = U(t)(x) = \left(e^{tL}u_0\right)(x).$$

Chernoff tangency

Chernoff tangency conditions the following:

(CT0) Let $\mathcal{F}$ be a Banach space, and let $\mathscr{L}(\mathcal{F})$ be the space of all bounded linear operators on $\mathcal{F}$. Suppose a map $G\colon [0;+\infty) \to \mathscr{L}(\mathcal{F})$ is given;

(CT1) The family G is strongly continuous in strong operator topology of the space $\mathscr{L}(\mathcal{F})$, i.e., the map $t \longmapsto G(t)f \in \mathcal{F}$ is continuous on $[0;+\infty)$ for each $f \in \mathcal{F}$;

(CT2) $G(0) = I$;

(CT3) There exists a linear subspace $D \subset \mathcal{F}$ dense in $\mathcal{F}$ such that for each $f \in D$ the limit

$$\lim_{t \to +0} \frac{G(t)f - f}{t}$$

exists. We denote its value by $G'(0)f$;

(CT4) The closure of the operator $(G'(0), D)$ exists and is equal to $(L, Dom(L))$.

Chernoff theorem, summary

Chernoff's theorem is a theorem on the «second remarkable limit» for for C_0-semigroup:

Let $\mathcal{F}$ — be a Banach space and L — be a closed linear operator in $\mathcal{F}$ with a dense domain. Let a family $(G(t))_{t\geq 0}$ of linear bounded operators in $\mathcal{F}$. Let the conditions also be true::
(E) C_0-semigroup $\left(e^{tL}\right)_{t\geq 0}$ exists
(N) There is such $\omega \in \mathbb{R}$ that $\|G(t)\| \leq e^{wt}$ for each $t \geq 0$
(CT) idea of the condition briefly: $G(t)f = f + tLf + o(t), t \to 0$
Then $e^{tL}f = \lim\limits_{n\to\infty} G(t/n)^n f$ for each $f \in \mathcal{F}$ and for each $t \geq 0$.
«second remarkable limit»

$$e^{tL} = \lim_{n\to\infty} G(t/n)^n = \lim_{n\to\infty} \left(I + \frac{tL}{n} + o(t/n)\right)^n$$

Theorem 1. Galkin-Remizov theorem (on estimation of speed of convergence of Chernoff approximations)

1. $T > 0$ is given, and C_0-semigroup $(e^{tL})_{t\geqslant 0}$ with generator $(L, D(L))$ in Banach space $\mathcal{F}$ satisfies for some $M_1 \geq 1$ and $w \geq 0$ the condition $\|e^{tL}\| \leqslant M_1 e^{wt}$ for all $t \in [0, T]$.
2. There is a mapping $G\colon (0, T] \to \mathscr{L}(\mathcal{F})$, i.e. $S(t)\colon \mathcal{F} \to \mathcal{F}$ is a linear bounded operator for each $t \in (0, T]$. There exists some constant $M_2 \geq 1$ that $\|G(t)^k\| \leqslant M_2 e^{kwt}$ for all $t \in (0, T]$ and all $k = 1, 2, 3, \ldots$
3. Numbers $m \in \{0, 1, 2, \ldots\}$ and $p \in \{1, 2, 3, \ldots\}$ are fixed. There is a $(e^{tL})_{t\geqslant 0}$-invariant subspace $\mathcal{D} \subset D(A^{m+p}) \subset \mathcal{F}$ (i.e. $(e^{tL})(\mathcal{D}) \subset \mathcal{D}$ for any $t \geq 0$, for example $\mathcal{D} = D(L^{m+p})$ is well suited).

4. There exist such functions $K_j \colon (0, T] \to [0, +\infty)$, $j = 0, 1, \ldots, m + p$ that for all $t \in (0, T]$ and all $f \in \mathcal{D}$:

$$\left\| G(t)f - \sum_{k=0}^{m} \frac{t^k L^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{m+p} K_j(t) \| L^j f \|.$$

Then:

1. For all $t > 0$, all integer $n \geq t/T$ and all $f \in \mathcal{D}$ we have
$$\|G(t/n)^n f - e^{tL} f\| \leq \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n)\|L^j f\|,$$
where $C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$ and $C_j(t) = K_j(t)e^{-wt}$ for $j \neq m+1$.
2. If $\mathcal{D}$ is dense in $\mathcal{F}$ and for all $j = 0, 1, \ldots, m+p$ we have $K_j(t) = o(t^{-m})$ when $t \to +0$, then for all $g \in \mathcal{F}$ and $\mathcal{T} > 0$ the following equality is true:
$$\lim_{\mathcal{T}/T \leq n \to \infty} \sup_{t \in (0, \mathcal{T}]} \left\| G(t/n)^n g - e^{tL} g \right\| = 0.$$

Theorem 2 (on estimation of norms of derivatives)

Suppose $n \in \{0, 1, 2, \dots\}$, the functions $a, b, c \colon \mathbb{R} \to \mathbb{R}$ are differentiable $2\lfloor (n-1)/2 \rfloor$ times and the inequality $\inf_{x\in\mathbb{R}} |a(x)| > 0$ holds. Suppose, in addition, the operator L maps each doubly differentiable function $u \colon \mathbb{R} \to \mathbb{R}$ to the function $Lu = au'' + bu' + cu$. Then there are nonnegative constants $C_0, C_1, \dots, C_{\lfloor (n+1)/2 \rfloor}$, such that for any $2\lfloor (n+1)/2 \rfloor$ times differentiable function $v \colon \mathbb{R} \to \mathbb{R}$, the following inequality is true:

$$\|v^{(n)}\| \leqslant \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} C_k \|L^k v\|.$$

Model equation and problem statement

For the next Cauchy problem

$$\begin{cases} \text{u'}_t(t,x) = a(x)u''_{xx}(t,x) + b(x)u'_x(t,x) + c(x)u(t,x) \\ \text{u(0,x)=u}_0(x) \end{cases}$$

We present the solution $u(t,x)$ in the form of a limit of fast converging Chernoff approximations under the conditions $f \in UC_b^8(\mathbb{R})$, $a, b, c \in UC_b^6(\mathbb{R})$, $\inf_{x\in\mathbb{R}} a(x) > 0$.

In this case $(Lf)(x) = a(x)f''(x) + b(x)f'(x) + c(x)f(x)$,

$$(L^2 f)(x) = a(x)^2 f^{(4)}(x) + (2a(x)a'(x) + 2a(x)b(x))f'''(x) + \\ + (a(x)a''(x) + 2a(x)b'(x) + 2a(x)c(x) + b(x)a'(x) + b(x)^2)f''(x) + \\ + (a(x)b''(x) + 2a(x)c'(x) + b(x)b'(x) + 2b(x)c(x))f'(x) + \\ + (a(x)c''(x) + b(x)c'(x) + c(x)^2)f(x)$$

Fast converging approximations to the solution of the model equation

Main result: we constructed the Chernoff function

$$(G_1(t)f)(x) = \frac{2}{3}f(x) + \frac{1}{6}f\left(x+\sqrt{6a(x)t}\right) + \frac{1}{6}f\left(x-\sqrt{6a(x)t}\right) +$$
$$+ a(x)a'(x)t\Big(3f\left(x+\sqrt[3]{t}\right) - 3f\left(x+2\sqrt[3]{t}\right) + f\left(x+3\sqrt[3]{t}\right)\Big) +$$
$$+\frac{1}{2}a(x)a''(x)t\Big(f\left(x+\sqrt{t}\right)+f\left(x-\sqrt{t}\right)\Big)-\Big(a'(x)+a''(x)\Big)a(x)tf(x),$$

that satisfies the condition

$$(G_1(t)f)(x) = f(x) + t(Lf)(x) + \frac{t^2}{2}(L^2 f)(x) +$$

$$\left(\frac{1}{18}a(x)a'(x)f^{(5)}(x) + \frac{1}{110}a(x)^2 f^{(6)}(x)\right)t^{5/2} + o(t^{5/2}).$$

Solution of the Cauchy problem

$$\begin{cases} \text{u'}_t(t,x) = a(x)u''_{xx}(t,x) + b(x)u'_x(t,x) + c(x)u(t,x) \\ \text{u(0,x)=u}_0(x) \end{cases}$$

is

$$u(t,x) = \lim_{n\to\infty}\left(G_1\left(\frac{t}{n}\right)^n u_0\right)(x).$$

Suppose $f \in UC_b^8(\mathbb{R})$, $a, b, c \in UC_b^6(\mathbb{R})$, $\inf_{x\in\mathbb{R}} a(x) > 0$.Then convergence speed estimate is

$$\left\| G_1(t/n)^n f - e^{tL} f \right\| \leq Ce^{\omega t} \frac{t^{7/3}}{n^{4/3}},$$

where $C \in \mathbb{R}$.

We also constructed an improved Chernoff formula

$$(G_2(t)f)(x) = \frac{2}{3}f(x)+\frac{1}{6}f\left(x+\sqrt{6a(x)t}\right)+\frac{1}{6}f\left(x-\sqrt{6a(x)t}\right)-$$
$$-a(x)a'(x)t\Big(\frac{7}{2}f\left(x+\sqrt[3]{t}\right)+\frac{1}{4}f\left(x-\sqrt[3]{t}\right)-\frac{7}{4}f\left(x+2\sqrt[3]{t}\right)+$$
$$+\frac{1}{4}f\left(x-2\sqrt[3]{t}\right)+\frac{1}{4}f\left(x+3\sqrt[3]{t}\right)\Big)+$$
$$+\frac{1}{2}a(x)a''(x)t\Big(f\left(x+\sqrt{t}\right)+f\left(x-\sqrt{t}\right)\Big)+\Big(\frac{5}{2}a'(x)-a''(x)\Big)a(x)tf(x)$$

that satisfies the condition

$$(G_2(t)f(x) = f(x) + t(Lf)(x) + \frac{t^2}{2}(L^2 f)(x) + R(x)t^3 + o(t^3),$$

where R - polynomials of the functions a, b, c and their derivatives of order to 2 and derivatives of function f of order to 8. Solution of the Cauchy problem

$$\begin{cases} \text{u'}_t(t,x) = a(x)u''_{xx}(t,x) + b(x)u'_x(t,x) + c(x)u(t,x) \\ \text{u(0,x)=u}_0(x) \end{cases}$$

is

$$u(t,x) = \lim_{n\to\infty} \left(G_2\left(\frac{t}{n}\right)^n u_0 \right)(x).$$

Suppose $f \in UC_b^8(\mathbb{R})$, $a, b, c \in UC_b^6(\mathbb{R})$, $\inf_{x\in\mathbb{R}} a(x) > 0$.Then convergence speed estimate is

$$\left\| G(t/n)^n f - e^{tL} f \right\| \leq C e^{wt} \frac{t^3}{n^2} + o\left(\frac{1}{n^2}\right),$$

where $C \in \mathbb{R}$.

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