On-line «Master in Computer Vision» educational program.
Entrance examination test (each problem is evaluated by 10 points)**.**

1. Find all solutions in integer numbers $x, y \in \{0,1,2,\dots,2020,2021\}$ of the equation

$$\frac{e^x}{e^x + e^y} = \frac{e^{2019}}{e^{2019} + 1}$$

2. There are given three row vectors of dimension 3:

$$u = (0, a, -1), \qquad v = (1, 1, a), \qquad z = (3, 5, 1)$$

Find all values of $a \in R$, such that the vector *z* is a linear combination of the vectors *u* and *v*.

3. Independent identically distributed discrete random variables *X* and *Y* have probability mass function (probability distribution) defined by the table

Values of X (or Y)	-2	1	2
Probability	1/5	2/5	2/5

Random variables *U* and *V* are defined by

$$U = 2X + Y, V = 2X - Y$$

Find expectation (expected value, mean) of the random variable $Z = U \cdot V$

4. In the following problem it is necessary to suggest the most efficient algorithms. Full points are given for the most efficient algorithm having the lowest computational complexity. The lower the efficiency of the suggested solution, the lower are the points.

For a given array of real numbers *a[i], i=0,1,2,…,n-1* and a given threshold $t \in R$, a *t*-chain in the array is a sequence of elements

a[k], a[k+1], a[k+2],…, a[k+l-1]

such that *a[k+j] ≥ t* for *j=0,1,…,l-1*.
Write a pseudo-code (or code on any programming language) of an algorithm, which takes as input an array of real numbers and a threshold *t* and gives as output the maximal length of *t*-chain in the array. Discuss the computational complexity of your algorithm.

1. Find all solutions in integer numbers $x, y \in \{0,1,2, \ldots ,2020,2021\}$ of the equation

$$\frac{e^x}{e^x + e^y} = \frac{e^{2019}}{e^{2019} + 1}$$

Solution

One has

$$e^x(e^{2019} + 1) = e^{2019}(e^x + e^y)$$

$$e^{x+2019} + e^x = e^{2019+x} + e^{2019+y}$$

$$e^x = e^{2019+y}$$

$$e^{x-y-2019} = 1$$

$$x - y - 2019 = 0$$

$$x = y + 2019 \Leftrightarrow y = x - 2019$$

Check possibilities for $x, y \in \{0,1,2, \ldots ,2020,2021\}$

$y = 0, x = 2019$

$y = 1, x = 2020$

$y = 2, x = 2021$

No other solutions.

Answer:

$y = 0, x = 2019$

$y = 1, x = 2020$

$y = 2, x = 2021$

2. There are given three row vectors of dimension 3:

$$u = (0, a, -1), \qquad v = (1, 1, a), \qquad z = (3, 5, 1)$$

Find all values of $a \in R$, such that the vector *z* is a linear combination of the vectors *u* and *v*.

Solution.

Vector *z* is a linear combination of the vectors *u* and *v* if there exist real numbers *t* and *s* such that

$$z = t \cdot u + s \cdot v$$

In coordinates it gives the system of equations for *t, s, and a*

$$\begin{cases} 3 = t \cdot 0 + s \cdot 1 \\ 5 = t \cdot a + s \cdot 1 \\ 1 = t \cdot (-1) + s \cdot a \end{cases}$$

It implies $s = 3$, and

$$\begin{cases} t \cdot a = 2 \\ 1 + t = 3a \end{cases}$$

One has $a \neq 0 \Rightarrow \ t = \frac{2}{a} \ \Rightarrow \ 1 + \frac{2}{a} = 3a \ \Rightarrow 3a^2 - a - 2 = 0 \Rightarrow \ a = 1 \ or \ a = -\frac{2}{3}$

We check both possibilities

$a = 1 \Rightarrow t = 2, \ z = 2 \cdot u + 3 \cdot v$, true

$a = -\frac{2}{3} \Rightarrow t = -3, \ z = (-3) \cdot u + 3 \cdot v$, true

Answer: $a = 1 \ and \ a = -\frac{2}{3}$

3. Independent identically distributed discrete random variables X and Y have probability mass function (probability distribution) defined by the table

Values of X (or Y)	-2	1	2
Probability	1/5	2/5	2/5

Random variables U and V are defined by

$$U = 2X + Y, V = 2X - Y$$

Find expectation (expected value, mean) of the random variable $Z = U \cdot V$

Solution.

One has $Z = U \cdot V = (2X + Y)(2X - Y) = 4X^2 - Y^2$

Therefore $E(Z) = E(4X^2 - Y^2) = 4E(X^2) - E(Y^2) = 3E(X^2)$

Probability mass distribution for X^2 is

Values of X^2	4	1
Probability	3/5	2/5

$$E(X^2) = 4 \cdot \frac{3}{5} + 1 \cdot \frac{2}{5} = \frac{14}{5}$$

It implies $E(Z) = 3E(X^2) = \frac{42}{5}$

Answer: $E(Z) = 8{,}4$

4. In the following problem it is necessary to suggest the most efficient algorithms. Full points are given for the most efficient algorithm having the lowest computational complexity. The lower the efficiency of the suggested solution, the lower are the points.

For a given array of real numbers *a[i], i=0,1,2,…,n-1* and a given threshold $t \in R$, a *t*-chain in the array is a sequence of elements

$$a[k], a[k+1], a[k+2], \ldots, a[k+l-1]$$

such that $a[k+j] \geq t$ for $j=0,1,\ldots,l-1$.

Write a pseudo-code (or code on any programming language) of an algorithm, which takes as input an array of real numbers and a threshold *t* and gives as output the maximal length of *t*-chain in the array. Discuss the computational complexity of your algorithm.

```
nput: array a, t
bool in_chain = false
l_max = 0
l = 0
for i = 0 to n - 1
if a[i] >= t
l += 1
in_chain = true
endif
else:
if in_chain == true:
in_chain = false
if l > l_max:
l_max = l
endif
l = 0
endif
endfor
return l_max
```

the number of computations in this algorithm can be calculated like this: We do operations on n elements only once, every time we check if a[i] is greater or equal to t(1), if so, we do 2 operations(incrementation of l and setting in_chain to true). Otherwise we do 1 check for in_chain to be true(1), if it is so, then set in_chain to false(1) and do a check for l being greater than l_max, calculated previously. If it is the case, we do 2 assignments. Since there is only 1 loop of n operations in this algorithm, the complexity of this algorithm is O(n)