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Contents

Periodic Solutions of the Generalized Mackey–Glass Equation with n Delays

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Mackey–Glass equation was first considered in [1] and was used to describe the concentration of blood cells. It is a differential equation with delay

$$\frac{du}{dt} = -\beta u + \frac{\alpha u(t-h)}{1+(u(t-h))^\gamma}. \tag{1}$$

Here parameters α, β, γ, h and unknown function $u(t)$ are positive.

We consider a generalization of the Mackey–Glass equation:

$$\frac{du}{dt} = -\beta u + \frac{\alpha\big(u(t-h_1)+\ldots+u(t-h_n)\big)}{1+\Big(\big(u(t-h_1)+\ldots+u(t-h_n)\big)\Big)^\gamma}. \tag{2}$$

Here n is natural, $0 < h_1 < \ldots < h_n$. The difference from the traditional Mackey–Glass equation (1) is the number of delays.

For a generalized equation, we consider the case of the large parameter γ. As γ tends to infinity, the function $\frac{1}{1+w^\gamma}$ on the right side of the differential equation tends to the threshold function

$$F(w) \overset{\text{def}}{=} \lim_{\gamma\to+\infty} \frac{1}{1+w^\gamma} = \begin{cases} 1, & 0 < w < 1, \\ \frac{1}{2}, & w = 1, \\ 0, & w > 1. \end{cases}$$

As a limit object for (2), we obtain the equation

$$\dot{u} = -\beta u + \alpha w F(w), \tag{3}$$

where $w(t) = u(t-h_1) + \ldots + u(t-h_n)$.

We study the question of the existence and stability of a periodic solution of the equation (3).

This work was supported by the Russian Science Foundation (project No. 21-71-30011)

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On bifurcations of solutions of impulsive systems

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In this report, we will try to give an overview of the current state of the theory of bifurcations of solutions to impulsive systems.

Many evolution processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Impulsive differential equations [1, 2] are powerful tools to model this type of evolution processes. Impulsive differential equations see applications in numerous fields where the systems of study exhibit rapid jumps in state. Such jumps may be intrinsic to the system, such as in the firing of a neuron in a biological neural network, or synthetic, such as the application of an insecticide or antibiotic treatment in a biological model. One of the most common applications of the theory of impulsive differential equations arises in the case, where a continuous autonomous system is perturbed by impulses in an impulsive control setting [3, 4]. Despite the large number of publications devoted to impulsive systems of the considered type [5]- [8], the qualitative theory, in particular, the theory of bifurcations is far from being sufficiently developed. Only recently there have appeared very meaningful studies devoted to this direction [9]- [11].

Mathematical models are considered, the results of computer modeling and examples of applications are given.

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Application of the classical theory of solitons to study the dynamics in passive-dispersed and active-relaxation media during the formation of twist in a toroidal plasma

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This paper presents a dynamic classical soliton model of the conditions for the formation of a swirl in a toroidal vortex of a stellarator. It is represented by the geometric construction of the evolution of the Smale-Williams hyperbolic attractor [1,2]. Two coupled Van der Pol generators are used as the main model $q_{n+1} = 2q_n$. Based on non-linear dynamics, the Bernoulli map represents the evolution of the torus. The use of the dynamic classical soliton model makes it possible to find the conditions for the formation of a swirl in a toroidal vortex.

Mathematical dynamic model of a soliton for a toroidal vortex

The mathematical classical dynamic model of the soliton is represented by the equation [3]:

$$\begin{cases} \ddot{x} - (A_1 \cos\omega t - x^2)\dot{x} + \omega_0^2 x = \varepsilon_1 y \cos\omega_0 t, \\ \ddot{y} - (-A_2 \cos\omega t - y^2)\dot{y} + \omega_0^2 y = \varepsilon_2 x^2, \\ u = xy \sinh x \sinh y \sin\omega_1 t \cos\omega_1 t. \end{cases}$$

Here x, y, u - dynamic variables, A_1, A_2, ε_1 and ε_2 - coefficient of connection, ω, ω_0 and ω_1 -inherent frequency oscillations.

The use of this hyperbolic model translates into practical implementation. This system can be used to study toroidal vortex processes with the formation of a stellarator twist.

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Shilnikov attractors in a minimal network of coupled neuron models

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In the work we consider minimal ensemble of Hodgkin-Huxley-type of neuron models. Typical for such systems is that the anti-phase synchronization is suspected here. In works [1-3], it was shown that, taking into account various types of coupling between subsystems (both excitatory and inhibitory), it is possible to single out in-phase synchronization. It is also shown that multistability between in-phase and anti-phase oscillatory modes is possible in this system. Also such models can demonstrate in-phase and anti-phase synchronous bursting behavior. Both bursting and spiking attractors can be chaotic. We analyze chaotic dynamics and reveal different types of chaos including hyperchaos. We study in detail route to hyperchaos which associated with the torus destruction and formation Shilnikov attractors [4].

As a base for analysis we use Hodgkin-Huxley-type models: Hindmarsh-Rose model and Sherman model. We consider different parameters of sub-systems, when they demonstrate bursting oscillations of different type, or spiking oscillations. We analyse parameter space for different type of coupling (inhibitory and excitatory). We have shown the formation of Shilnikov chaotic attractors which leads to the appearance hyperchaos [5].

The work is supported by the grant of Russian Science Foundation (Project No. 20-71-10048).

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On Soft Loss of Stability in Ocean Circulation Box Model with Turbulent Fluxes

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We consider a two dimensional system of ordinary differential equations that provides a qualitative description of the thermohaline circulation locally in the upper layer of water in ocean [1], [2], [3]. We show that in a generic finite parametric case of transfer functions from the system the soft loss of stability of a steady state in this system is described by the appearance of nonzero fixed points of composition of pair of two involutions of the real line with fixed point at zero, which are defined by the system. The respective bifurcation diagrams in the space of parameters are also described.

The study was financially supported by the Russian Science Foundation, project no. 19-11-00223

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Nonlinear dynamics and typical bifurcations in the model of three coupled ultrasound contrast agents

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We study dynamics in the model describing oscilaltions of three coupled encapsulated microbubbles under the influence of external pressure field [1, 2]. We study three configuration of bubbles: in the vertices of an equilateral triangle, in the vertices of an isosceles triangle and in the nodes of a chain with equal distances between the neighboring bubbles.

In the symmetrical case of equilateral triangle, the dynamics can be fully synchronous, partially synchronous, or asynchronous. In the other two spatial configurations it could be either partially synchronous or asynchronous. We find that the main mechanism of destruction of synchronization in the completely symmetrical configuration is the bubbling transition scenario. An interesting feature of the implementation of this scenario in this model is that the asynchronous component of the trajectory has three positive Lyapunov exponents. On the other hand, the partial synchronization in case of the chain or an isosceles triangle configurations, is usually destroyed via a special case of period-doubling bifurcation, when the initial limit cycle becomes unstable in the direction transversal to the partial synchronization hyperplane.

We show that fully synchronous or partially synchronous chaotic attractors usually emerge via a period-doubling cascade. While asynchronous chaotic regimes emerge either via a period–doubling cascade of an asynchronous limit cycle or the Afraimovich–Shilnikov scenario, starting from a partially synchronous limit cycle. Hyperchaotic attractors with two positive Lyapunov exponents typically occur via the scenario involving emergence of the discrete Shilnikov attractor [3]. We also observed hyperchaotic attractors with three positive Lyapunov exponents in different spatial configurations, although the details of scenario of their emergence are not yet clear.

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On rationally integrable planar dual and projective billiards

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A *caustic* of a strictly convex planar bounded billiard is a smooth curve whose tangent lines are reflected from the billiard boundary to its tangent lines. The famous Birkhoff Conjecture states that if the billiard boundary has an inner neighborhood foliated by closed caustics, then the billiard is an ellipse. It was studied by many mathematicians, including H.Poritsky, M.Bialy, S.Bolotin, A.Mironov, V.Kaloshin, A.Sorrentino and others.

We study its following generalized *dual* version stated by S.Tabachnikov. Consider a closed smooth strictly convex curve $\gamma \subset \mathbb{RP}^2$ equipped with a *dual billiard structure:* a family of non-trivial projective involutions acting on its projective tangent lines and fixing the tangency points. *Suppose that its outer neighborhood admits a foliation by closed curves (including γ) such that the involution of each tangent line permutes its intersection points with every leaf. Then γ and the leaves are conics forming a pencil.*

We prove positive answer in the case, when the curve γ is C^4-smooth and the foliation admits a rational first integral. To this end, we show that each C^4-smooth germ γ of planar curve carrying a rationally integrable dual billiard structure is a conic and classify all the rationally integrable dual billiards on conics. They include the dual billiards induced by pencils of conics, two infinite series of exotic dual billiards and five more exotic ones.

Determinantal processes and decomposition of functions into series defined by values in points of a random configuration

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Determinantal processes is a class of random point fields, that is, probability measures on a set of discrete subsets (or *configurations*) of some phase space E, which show a mix of random and deterministic behavior.

A determinantal process is defined by a contraction operator on the space $L^2(E)$. In most known examples this operator is an orthogonal projection onto some subspace $H \subset L^2(E)$, which consists of sufficiently regular functions, so that one can define the values of a function $f \in H$ in each point of the space E. This allows us to close the loop between the measure on the space of configurations and the subspace H: is $f \in H$ uniquely defined by its values on X, for almost all configurations X?

This is known to be true for a wide class of determinantal processes, and moreover, as A. Bufetov [1] has shown, there is a constant $k \geq 0$, which is called *an excess* of the process, such that for almost any configuration X and any choice of k points $x_1, \ldots, x_k \in X$ a function $f \in H$ is uniquely defined by its values on $X \setminus \{x_1, \dots, x_k\}$, and if we remove any $(k+1)$ points from almost any configuration, there exists a function $f \in H$ that vanishes on $X \setminus \{x_1, \dots, x_{k+1}\}$.

We are dealing with a more delicate question: is it possible to reconstruct a function f from its values on $Y = X \setminus \{x_1, \dots, x_k\}$? This problem is linear, so one can start with functions g_s such that $g_s(y) = \delta_{s,y}$ for $s, y \in Y$. Then one can expect that

$$f(x) = \sum_{s \in Y} f(s) g_s(x) \quad \text{for all } x \in E.$$

Both parts agree for $x \in Y$, so the identity holds, provided that the series in converging. We have shown that for some determinantal processes this series does converge in $L^2(E)$ for a functions f from a finite-codimension subspace of H.

The talk is based on a joint work in progress with Alexander Borichev and Alexander Bufetov.

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Zeta Invariants of Morse Forms

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Given a closed real differential 1-form η on a closed Riemannian manifold (M, g) of dimension n, let

$$d_z = d + z\,\eta\wedge, \quad \delta_z = \delta - \bar{z}\,\eta\lrcorner, \quad \Delta_z = d_z\delta_z + \delta_z d_z, \quad z \in \mathbb{C},$$

be the induced Witten's type perturbations of the de Rham derivative d, de Rham coderivative $\delta = d^*$ and Hodge Laplacian $\Delta = d\delta + \delta d$, respectively, on the space $\Omega(M)$ of differential forms on M. We introduce the zeta function $\zeta(s, z)$ by the formula

$$\zeta(s, z) = \sum_{k=1}^{n} (-1)^k \operatorname{Tr}(\eta\wedge\, \delta_z \Delta_z^{-s} : \Omega^k(M) \to \Omega^k(M))$$

which is well defined for $\Re s \gg 0$ and $z \in \mathbb{C}$. As a function of s, it is holomorphic for $\Re s \gg 0$ and admits the meromorphic extension to $\mathbb{C}$.

Assuming that

(a) η is a Morse form (it has Morse type zeros), and g is Euclidean with respect to Morse coordinates around the zero points of η,

we prove that $\zeta(s, z)$ is smooth at $s = 1$ for $|\Re z| \gg 0$.

We study the asymptotic behavior of the zeta invariant $\zeta(1, z)$ as $|\Re z| \to \infty$. We assume that there exists an auxiliary vector field X such that:

(b) X has Morse type zeros, and is gradient-like and Smale;

(c) η is Lyapunov for X, and η and g are in standard form with respect to X;

(d) for every zero point p of X, the maximum value of the integrals of η along the instantons of X with α-limit $\{p\}$ only depends on the Morse index k of p.

It is shown that, for any vector field X satisfying (b) and for any cohomology class in $H^1(M,\mathbb{R})$, there is a representative η of this class and a metric g satisfying (a), (c) and (d).

Under current assumptions on η and g, we prove that the zeta invariant $\zeta(1,z)$ converges to some $\mathbf{z} \in \mathbb{R}$ as $\Re z \to +\infty$, uniformly on $\Im z$. The limitting value $\mathbf{z}$ is described in terms of the instantons of the vector field X and the Mathai-Quillen current on TM defined by g.

If n is even, we can prescribe any real value for $\mathbf{z}$ by perturbing g, η and X; moreover we can also achieve the same limit as $\Re z \to -\infty$.

These results are used to define and describe certain tempered distributions on the real line induced by g and η. These distributions appear in [2] as the contributions from the compact leaves preserved by the flow in a trace formula for simple foliated flows on closed foliated manifolds, which gives a solution to a problem proposed by C. Deninger (see, for instance, [3]).

This is joint work with J.A. Álvarez López and E. Leichtnam [1].

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Graphs of Dynamical Systems on Surfaces and Efficient Algorithms for Their Distinguishing

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I was privileged, and I am honoured to be a coauthor of Prof. O.V. Pochinka in the papers [1]–[6]. They are devoted to combinatorial (predominantly, graph) invariants of some dynamical systems classes and efficient (mostly, polynomial-time) algorithms for solving the isomorphism problem for them. My talk will be devoted to these results, which will be dedicated to 50th-anniversary of O.V.

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Approximative properties of the invariant norms determined by the orbitopes of the polar representations of the compact simple Lie groups

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According to [1], the group of Hamiltonian diffeomorphisms Ham (M, ω) of a compact symplectic manifold M has a natural Finsler metric, called the Hofer metric, whose geodesic geometry reflects various dynamical properties of Hamiltonian vector fields. The Lie algebra of this group consists of Hamiltonian fields, and the norm of a vector field is $L^\infty(M)$ - norm of its Hamiltonian. In the situation of a Hamiltonian action of a semisimple compact Lie group K on (M, ω), the Hofer metric induces a biinvariant the Finsler metric on the compact group K and the generalized Hofer norm on its Lie algebra $\mathfrak{K}$, which is invariant under the adjoint action of K on $\mathfrak{K}$. The work [2] is devoted to such a study. On the other hand, the properties of the class of unitarily invariant norms on the classical matrix algebras are actively studied in matrix analysis (see, for example, [3]).

In [4] E. B. Vinberg obtained a description of the class of invariant norms on simple compact Lie algebras in terms of norms invariant under the action of Weyl group on the Cartan subalgebras of these Lie algebras.

An interesting class of such finite-dimensional norms is obtained by taking as their unit balls the convex hulls of the orbits of the action of the Weyl group on the Cartan subalgebras. Polytopes of this kind turn out to be quite useful combinatorial invariants in the study of dynamical properties of Hamiltonian vector fields, defining a special class of Hofer metrics.

The above class of biinvariant Finsler metrics on compact groups K, independently arose in a number of problems about geodesics on compact homogeneous Riemannian and Finsler spaces (see [5]). As part of the study of the geometry of non-compact symmetric spaces equipped with invariant Finsler metrics, in [6] pointed out how the properties of convexity of unit balls of invariant norms affect on the number of properties of geodesics of these metrics.

In this report our aim is to analyze the combinatorial and approximative properties of the invariant norms determined by the orbitopes of the polar representations of the compact groups K.

We obtained a constructive description for the class of all linear subspaces with the property of uniqueness of the best approximation for the indicated class of invariant norms. Our results generalize some of the results of [7], where solved the problem of the best Chebyshev approximation with respect to the spectral norm on the space of square matrices. In addition, the connection between these questions and the well-known topological problem on the number of independent vector fields on spheres was also established there.

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Graded Lie algebras and critical points of the Ricci

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The affine variety L_n of Lie algebra products on a fixed n-dimensional vector space V consists of skew-symmetric bilinear maps $\mu : V \wedge V \to V$ satisfying the Jacobi identity. One can define the affine subvariety N_n of nilpotent Lie algebras [4]. We fixe rthonormal basis $e_1, \dots, e_n$ in V with respect to some Euclidian scalar product on V [1, 2].

The Ricci curvature operator R_μ for a nilpotent Lie algebra product $\mu \in N_n$ is defined by

$$R_\mu = \frac{1}{4}\sum_{i=1}^{n} ad_{e_i} ad^*_{e_i} - \frac{1}{2}\sum_{i=1}^{n} ad^*_{e_i} ad_{e_i}, \quad adX(Y) = \mu(X,Y).$$

Consider a function F on the variety N_n of nilpotent Lie algebras $F(\mu) = tr\ R^2_\mu$. In coordinates $c_{ijk} = g_{is}c^s_{jk}, [e_j, e_k] = c^s_{jk}e_s$ it can be written

$$F(\mu) = trR^2_\mu = \sum_{pr}\left(\sum_{ij}\left(-\frac{1}{2}c_{pij}c_{rij} + \frac{1}{4}c_{ijp}c_{ijr}\right)\right)^2, \ \mu = \{c_{ijk}\} \in N_n,$$

Lauret proved [1, 2] that if a nilpotent Lie algebra law $\mu \in N_n, ||\mu|| = 1$ is a critical point of the functional F then $L = (V, \mu)$ is a positively graded Lie algebra $L = \oplus_{i=1}^{+\infty} L_i$. Its orbit with respect to the GL_n-action on N_n can be considered as the isomorphism class of L

It was shown [1, 3] that in low dimensions ≤ 6 every GL_n-orbit in N_n contains a critical point of F. Lauret also considered a gradient flow defined by the function F.

We will discuss the relations of this techniques to the deformation theory of Lie algebras.

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On New Results in the Restricted Problem of Three Magnetic Vortices

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This work is a continuation of the study of the *restricted problem of three magnetic vortices* [1]. We consider vortices at positions $r_\alpha = (x_\alpha, y_\alpha)$, $\alpha = 1, \ldots, N$, with constant vorticities Γ_α and polarities λ_α, which take values ± 1 depending on a direction of magnetization. The restriction consists in fixing one of the vortices at the origin. In general, this model describes a motion of point vortices in ferromagnets [2]. In particular case, when $\lambda_\alpha = \lambda_\beta \ \forall \alpha, \beta$, the model also describes the motion of vortices in an ideal fluid [3]. This system is Hamiltonian:

$$H = \frac{\Gamma_1}{\lambda_1} \ln |r_1| + \frac{\Gamma_2}{\lambda_2} \ln |r_2| + \frac{\Gamma_1}{\lambda_1} \frac{\Gamma_2}{\lambda_2} \ln |r_1 - r_2|,$$

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \quad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \quad \alpha = 1, 2, \tag{1}$$

with an integral of the *angular momentum of vorticity* $F = \Gamma_1 r_1^2 + \Gamma_2 r_2^2$. Thus, it is a completely Liouville integrable system with two degrees of freedom.

The *bifurcation diagram* Σ of the momentum map $\mathcal{F}(\boldsymbol{x}) = (F(\boldsymbol{x}), H(\boldsymbol{x}))$ was found in a case of positive vorticities $\Gamma_1, \Gamma_2 > 0$ and positive polarities $\lambda_1 = \lambda_2 = 1$. We denote the vorticity ratio $\gamma = \frac{\Gamma_2}{\Gamma_1}$. In the case of $\gamma = 1$, the bifurcation diagram (Fig. 1) contains bifurcation $3\mathbb{T}^2 \to \mathbb{S}^1 \times \left(\mathbb{S}^1 \dot\cup \mathbb{S}^1 \dot\cup \mathbb{S}^1\right) \to \mathbb{T}^2$. Here $\mathbb{T}^2$ denotes the presence of a two-dimensional Liouville torus in the preimage of the momentum map. Such a bifurcation was found in another generalized vortex model [4] and in the integrable case of Goryachev-Chaplygin-Sretensky in the dynamics of a rigid body [5]. For a small perturbation in the parameter γ the critical integral surface is unstable and separates into two disconnected critical integral manifolds $\mathbb{S}^1 \times \left(\mathbb{S}^1 \dot\cup \mathbb{S}^1\right) \cup \mathbb{T}^2$ and $\mathbb{S}^1 \times \left(\mathbb{S}^1 \dot\cup \mathbb{S}^1\right)$. This fact is confirmed by the work on the splitting of saddle singularities [6].

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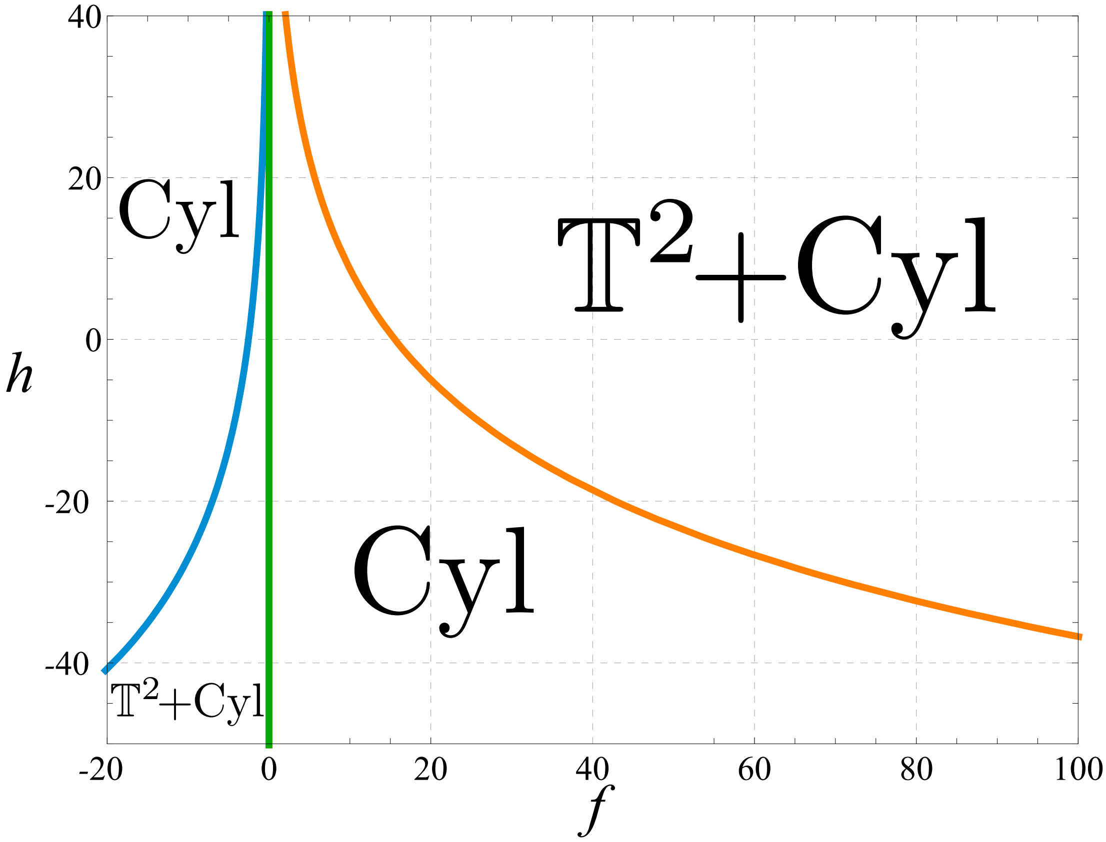

Figure 1: Bifurcation diagrams for different values of γ.

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Inverse Shadowing and Invariant Measures

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In this talk, we discuss various relations between the inverse shadowing property of discrete dynamical systems on a smooth closed manifold and invariant measures.

First, we introduce the so-called ergodic inverse shadowing property (Birkhoff averages of continuous functions along an exact trajectory and the approximating one are close). We demonstrate that this property implies the continuity of the set of invariant measures in the Hausdorff metric. We show that the class of systems with ergodic inverse shadowing is quite broad; it includes all diffeomorphisms with hyperbolic nonwandering sets.

Second, we study the so-called individual inverse shadowing (any exact trajectory can be traced by approximate ones, but this shadowing is not uniform with respect to the initial point of the trajectory). We demonstrate that, in terms of invariant measures, this property is closely related to structural stability and Ω-stability of diffeomorphisms.

The main results are published in [1].

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Remark on Resolvents of Degrees of Linear Operators

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Let V is Banach space and $A : V \to V$ is a linear operator. Further let one suppose that the resolvent operator $R_\lambda(A) = (A - \lambda\, I)^{-1}$ for A is known explicitly. The aim of this report is to show how to calculate the resolvent operator $R_\lambda(A^m) = (A^m - \lambda\, I)^{-1}$ for arbitrary natural degree m of operator A using expression for $R_\lambda(A)$. Examples demonstrating this procedure for both bounded and unbounded operators have been done.

Review research of the bifurcation analysis of the model of a Lagrange top with a vibrating suspension point

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The present report is review research of the bifurcation analysis of the mechanical model of a Lagrange top with a vibrating suspension point. Mechanical system describes the dynamics of a rigid body in a uniform gravity field. One of the points of the body lying on the axis of symmetry (the suspension point) performs high-frequency vertical oscillations of small amplitude. System of differential equations is a completely Liouville-integrable Hamiltonian system with two degrees of freedom. Such a system can be subjected to bifurcation analysis and clearly demonstrate the problems of stability research based on the analysis of the type of singularities. The type of rank-zero singularities of the integral mapping, which are associated with equilibrium positions, are determined in the paper [1]. In contrast to the classical approach used for stability analysis of the upper equilibrium in [2], an analysis of the type of singularities of the integral mapping revealed relations under which the lower equilibrium position becomes unstable. Additionally, a unique phenomenon is observed in the considered mechanical system, namely, the realization of doubly pinched torus. The bifurcation diagram and the atlas of bifurcation diagrams are explicitly defined. The problem of stability of regular precessions is carried out on the basis of determining the type of singularity and geometric interpretation of stability on the bifurcation diagram. The regular precessions will be unstable for the branch of the bifurcation curve between the cusped points. At the remaining points of the bifurcation curve, the regular precessions have an elliptical type, which corresponds to the stability of regular precessions.

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On branching of periodic solutions of linear inhomogeneous differential equations with two small parameters

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The work is devoted to the construction of a T-periodic solution of a linear inhomogeneous differential equation

$$A\frac{dx}{dt} = (B_0 - \varepsilon_1 B_1 - \varepsilon_2 B_2)\, x - f(t), \tag{1}$$

here $x \in E_1$; A, B_0, $B_{1\ 2}$ – densely defined linear Fredholm operators acting from E_1 to E_2; $f \in E_2$, $f(t+T) = f(t)$, $T > 0$; ε_1, ε_2 – small real parameters; A – degenerate or identical operator.

Here the desired T-periodic solution $x(t, \varepsilon_1, \varepsilon_2)$ to equation (1), must satisfy the condition $x(t, 0, 0) = z(t)$, where $z(t)$ — T-periodic solution of the equation [1]

$$A\frac{dz}{dt} = B_0 z - f(t). \tag{2}$$

Let the numbers $\pm i\alpha_m$ ($\alpha_m = k_m\omega$, $\omega = \frac{2\pi}{T}$, $k_m \in N$, $m = \overline{1,r}$) are the A-eigenvalues of the operator B_0; u_{mj} are A-eigen elements of the operator B_0

$$B_0 u_{mj} = i\alpha_m A u_{mj}, \quad B_0 \overline{u}_{mj} = -i\alpha_m A \overline{u}_{mj},$$

$j = \overline{1,n_m}$, $m = \overline{1,r}$; n_m – geometric multiplicity of each number in a pair $\pm i\alpha_m$ (the number of A-eigen elements that correspond to an eigenvalue $i\alpha_m$).

To solve this problem, a modified Lyapunov-Schmidt method is used [2]. It is based on construction of a complete generalized Jordan collection in the sense of the works [3, 4].

It is shown in the work that equation (1), provided that the point $(\varepsilon_1, \varepsilon_2)$ belongs to a sufficiently small punctured neighborhood of zero, has a unique Tperiodic analytic in ε_1 and ε_2 solution. Moreover, for $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$ the solution to equation (1) goes over into the $2n$parametric family of Tperiodic solutions to equation (2).

If the side lengths of all generalized Jordan grids are equal to one, then equation (1) has a unique Tperiodic solution that is analytic in ε_1 and ε_2 from a sufficiently small neighborhood of zero. Moreover, the solution of the perturbed equation (1) tends to the solution of the unperturbed equation (2) as ε_1 and ε_2 tend to zero.

If there is at least one generalized Jordan grid, the side length of which is greater than one, then the solution to equation (1) has a pole either in ε_1, or in ε_2, or simultaneously in ε_1 and ε_2.

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Extreme States of Interacting Solitons

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Interacting solitons (soliton gas or soliton turbulence) represent the opposite limit of interacting linear waves which possess the Gaussian statistics. It is in the focus of current research related to the problem of extreme phenomena (including the so-called rogue waves) in irregular waves, when the fraction of coherent states cannot be neglected. Due to the inherent nonlinearity, the theoretical description of a soliton gas in the general formulation is difficult and is still an open problem [1]. The kinetic description of ensembles of solitons cannot provide the requested information about the wave amplitudes.

Simplified model problems are considered as a constructive approach to solving the problem. In particular, in a rarefied soliton gas, it is usually sufficient to take into account only pairwise soliton interactions. Collisions between pairs of solitons were understood in [2] as "elementary acts" of soliton turbulence. Indeed, the picture of pair interactions agrees qualitatively with the results of direct numerical simulations of large ensembles of solitons. At the same time, multiple soliton collisions were also observed in numerical simulations [3, **?**]. Due to the strong nonlinearity, these situations are difficult for the direct numerical simulation [5]. However, they can lead to the generation of extremely high waves when solitons have different signs [6, 7].

Solitons of the same sign (as in the classical Kortweg–de Vries equation) do not form larger waves during interaction, but can form states with a critical (maximum) density [1, 8]. In this paper, we discuss some recent own results on the study of soliton states with a critical density. The exact solutions of the Korteweg–de Vries equation constructed using the original numerical subroutine are investigated. Surprisingly, some key characteristics of such soliton fields can be calculated analytically. These results should help further understanding of extreme soliton states and, eventually, the development of an appropriate theoretical description.

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Внешние бильярды относительно правильных многоугольников

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Внешние (или двойственные) бильярды относительно выпуклых фигур, введенные Б. Нойманом в 1950-ых, получили распространение благодаря интерпретации Ю. Мозера, согласно которой они служат упрощенными моделями для задач небесной механики. С другой стороны, внешние бильярды относительно многоугольников составляют частный случай (евклидовых) перекладываний. В работах С. Табачникова, Р. Шварца, Н. Бедарида, Ж. Кассэня и др. рассматриваются двойственные бильярды вне правильных n-угольников. Единственными нетривиальными, но хорошо понятыми случаями долго оставались случаи n=5, 10, описание которых основано на евклидовом самоподобии. В диссертации Ф. Руховича с использованием компьютерных вычислений описаны случаи n=8, 12. Мы рассмотрим случай 7-угольника, в котором были недавно обнаружены два принципиально разных самоподобия. Коэффициенты этих самоподобий порождают подгруппу полного ранга (=2) в группе единиц алгебраического поля – поля разложения многочлена $z^7 - 1(,)$....

Morse-Bott energy function for topological flows with a hyperbolic chain-recurrent set consisting of a finite number of orbits

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Regular topological dynamical systems are defined as dynamical systems whose chain-recurrent set is topologically hyperbolic and consists of a finite number of fixed points and periodic orbits. For such systems, provides an exhaustive description of the behavior of invariant manifolds of chain components, both from the point of view of asymptotics and from the point of view of the topology of their embedding in the carrier manifold.

Also it is proved that for a regular flow without periodic orbits, given on a topological manifold of any dimension, there exists a (continuous) Morse energy function. The result obtained is an ideological continuation of the work of S. Smale [1], in which he established the existence of a smooth energy Morse function for any gradient-like flow on a manifold, and a partial solution of the Morse problem on the existence of continuous Morse functions on any topological manifolds. Namely, a topological manifold admits a continuous Morse function if and only if it admits a regular topological flow without periodic orbits. This result was obtained in the present work within the framework of constructing a continuous Morse-Bott energy function for an arbitrary continuous regular flow on a topological manifold, and is an analogue of the theorem K. Meyer [2], who in 1968 constructed the Morse-Bott energy function for an arbitrary Morse-Smale flow on a smooth closed n-manifold.

We proves the existence of a continuous energy function for any regular flow. This is result are the ideological continuation of the works of S. Smale [1] and K. Meyer [2] on the existence of the Morse energy function for gradient-like flows and the Morse-Bott energy function for Morse-Smale flows, respectively.

A function φ is called a *continuous Morse-Bott function* if any connected component of the set Cr_φ is either a non-degenerate critical point or belongs to a non-degenerate critical submanifold.

Statement. *Any regular topological flow $f^t : M^n \to M^n$ without periodic orbits has a continuous energy Morse function.*

The concept of a continuous Morse function was introduced by Morse back in 1959 in [4], at the same time the validity of the Morse inequalities was proved for it, and later (in [5]) a number of properties similar to the properties of the smooth Morse function . However, the question of the existence of a continuous Morse function on an arbitrary topological manifold is still an open question. Since the continuous Morse function generates a topological gradient-like flow on the manifold [5], then Statement is a partial solution of the Morse problem: a topological manifold admits a continuous Morse function if and only if it admits a topological flow with a finite hyperbolic chain-recurrent set.

Statement follows directly from a more general result.

Theorem. *Any regular flow $f^t \in G^t$ has a continuous energy Morse-Bott function whose critical points are either non-degenerate or form non-degenerate one-dimensional manifolds.*

The existence of an energy function fundamentally distinguishes continuous regular systems from discrete ones. For the latter, an obstacle to the construction of the energy function is the possible presence of wild saddle separatrices discovered by D. Pixton [6] in 1977 in dimension three. Examples of regular flows with wild separatrices are also known; such flows are constructed, for example, in recent works by V. Medvedev and E. Zhuzhoma [7]. However, it follows from the results that for regular flows, the wildness of the separatrices is not an obstacle to the existence of the Morse energy function.

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