Высокоточная модель для проверки точности алгоритмов экспериментального определения кросскорреляции двух случайных процессов

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A high-precision model for verifying accuracy of algorithms for the experimental determination of the cross-correlation of two random processes

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Introduction

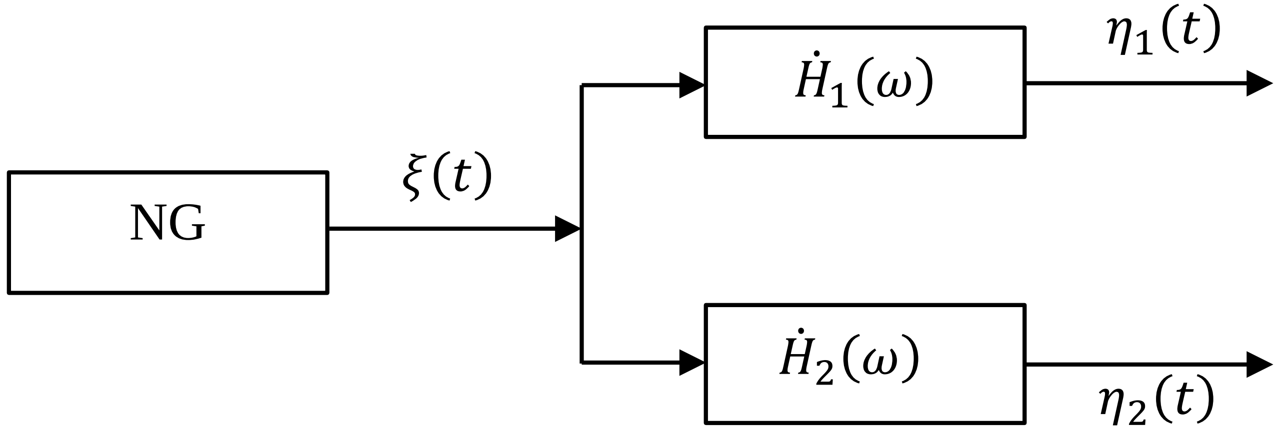
The correlation signal processing plays an important role in a variety of radar and radio communication tasks related to distinguishing useful signals against interference, optimal reception or determining the coordinates of objects based on the analysis of signals reflected from them. The current level of development of specialized computing technologies allows for creating plenty algorithms (including those based on the use of artificial intelligence and neural networks).

This paper considers an installation model that allows to obtain realizations of two random processes for which the cross-covariance is already known in an exact analytical form (and so can be used to verify the accuracy of functioning of algorithms performing experimental determination of the cross-covariance by given realizations; this includes artificial intelligence programming using the machine learning technologies).

Installation Description

The installation used for acquiring the random processes realizations $\eta_{1;2}(t)$ (cross-covariance of which can be exactly calculated, as it is shown below) consists of a centered wideband noise generator NG (for example, it can be implemented on a semiconductor Zener diode) and a two-channel linear filter made of two *LC*-circuits (with lumped elements).

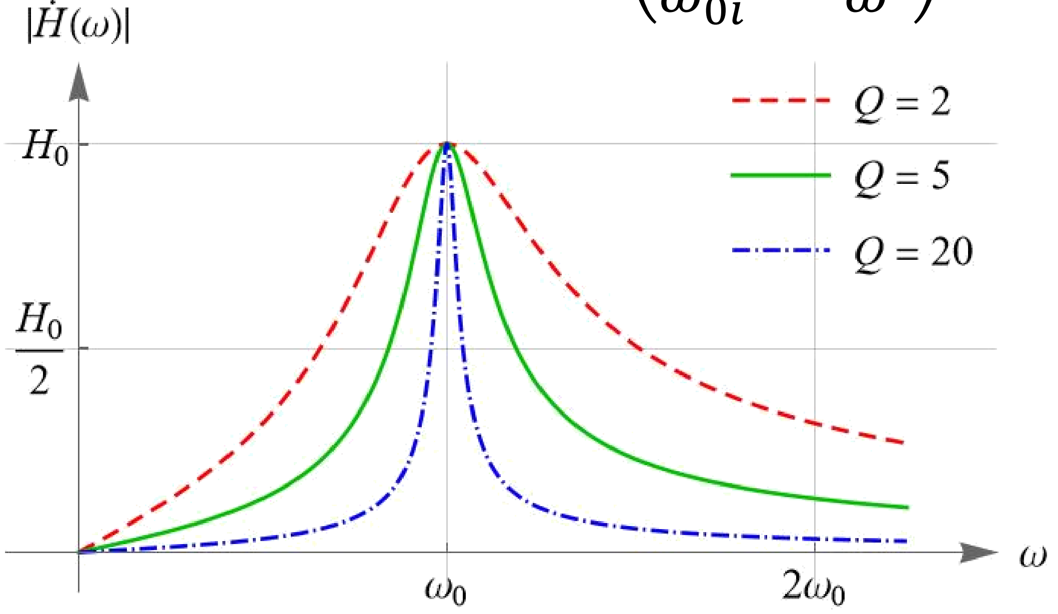
Note that we do not restrict the channels of the filter to be similar to each other or to satisfy any special criteria (such as ‘high quality factor’). In fact, their frequency responses $\dot{H}_{1;2}(\omega)$ are allowed to have a very general form, which is discussed on the next slide.

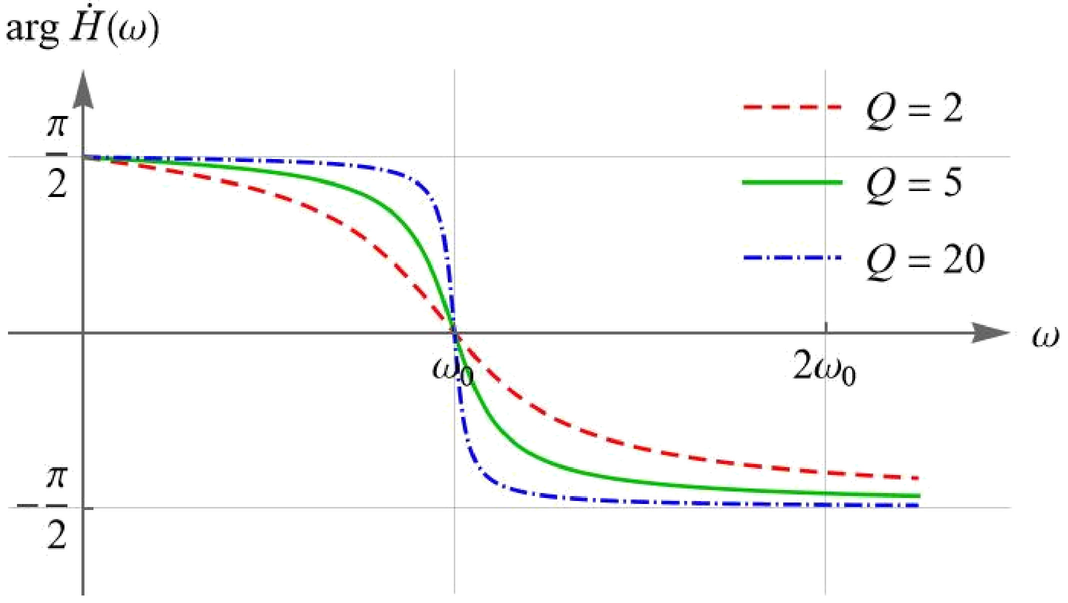


The Frequency Responses

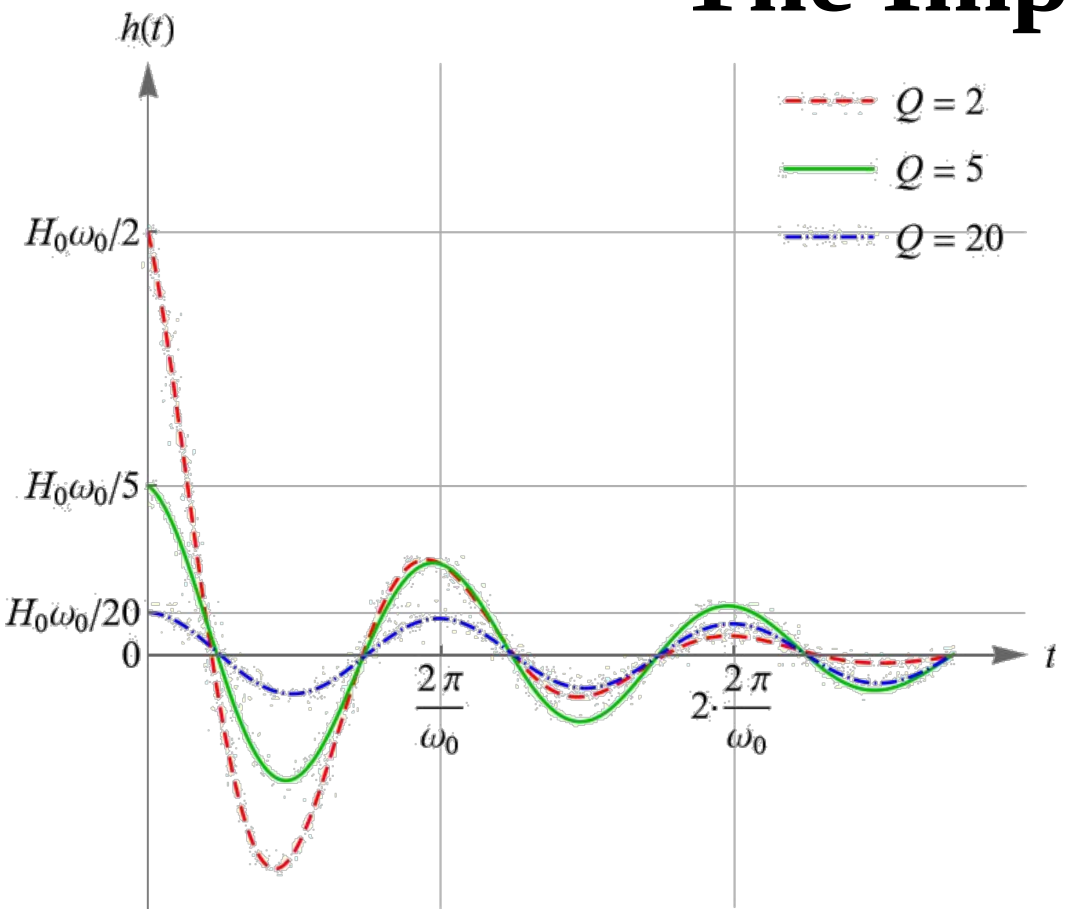
Each of the two channels of the filter, being a passive circuit, is characterized by its quality factor Q_i, the resonance frequency ω_{0i} and the value of the transfer coefficient on this frequency $H_{0i} = \dot{H}_i(\omega_{0i})$ (here index $i = 1; 2$ is used to distinguish the two circuits). The following form of the frequency response for such a linear system is well-known:

$$\dot{H}_i(\omega) = \frac{H_{0i}}{1 + jQ_i \cdot \left(\frac{\omega}{\omega_{0i}} - \frac{\omega_{0i}}{\omega}\right)}, \qquad Q_1 \not\equiv Q_2, \qquad H_{01} \not\equiv H_{02}, \qquad \omega_{01} \not\equiv \omega_{02}.$$





The Impulse Responses



$$h_i(t)=\frac{\Theta(t)H_{0i}\omega_{0i}}{Q_i}e^{-\frac{\omega_{0i}t}{2Q_i}}\left(\cos\widetilde{\omega}_i t-\frac{\sin\widetilde{\omega}_i t}{\sqrt{4Q_i^2-1}}\right)$$

The impulse response $h_i(t)$ of a circuit can be found as the inverse Fourier transform of $\dot{H}_i(t)$:

$$h_i(t)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{H_{0i}\omega_{0i}\cdot ze^{jzt}dz}{\omega_{0i}\cdot z+jQ_i\cdot\left(z^2-\omega_{0i}^2\right)};$$

Jordan's lemma allows us to close the integration contour by adding a semicircle arc $|z|=R$ located in the upper half-plane. Then we use the residue theorem (the integrand has two simple poles $z_k=\frac{j-(-1)^k\sqrt{4Q_i^2-1}}{2Q_i}\cdot\omega_{0i},\ k=1;2$).

Assuming that $Q_i>\frac{1}{2}$ (our only restriction on the circuits properties, almost always satisfied) and denoting $\widetilde{\omega}_i=\frac{\omega_{0i}}{2Q_i}\cdot\sqrt{4Q_i^2-1}$, we obtain the expression presented on the left (the occurrence of the Heaviside function $\Theta(t)$ corresponds to the circuits being causal linear systems).

Cross-Covariance: Expressing as an Integral

The proposed scheme produces random processes $\eta_{1;2}(t)$ which are both centered, stationary and ‘stationary correlated in broad sense’, and their cross-covariance $K_{12}(\tau) = \langle \eta_1(t+\tau)\eta_2(t) \rangle$ is expressed in terms of $h_i(t)$ and the auto-covariance of the generated noise $K_\xi(\tau) = \frac{N_0}{2}\delta(\tau)$:

$$K_{12}(\tau) = \int_0^\infty dv \int_0^\infty h_1(u)h_2(v)K_\xi(\tau - u + v)du = \int_0^\infty \frac{N_0}{2} h_1(\tau+v)h_2(v)dv\,.$$

Due to the symmetry of the cross-covariance as a function of its argument, it is enough to calculate it for $\tau > 0$ (what we assume from now on). Thus it is possible to substitute the impulse responses, taking into account that $\Theta(\tau+v) = 1 = \Theta(v)$ in the integrand. Introducing constants

$$K_0 = \frac{N_0 H_{01} H_{02} \omega_{01}\omega_{02}}{2Q_1Q_2}, \qquad \omega_0 = \frac{1}{2}\left(\frac{\omega_{01}}{Q_1} + \frac{\omega_{02}}{Q_2}\right), \qquad A_k = \sqrt{4Q_k^2 - 1},$$

one can get

$$K_{12}(\tau) = K_0 \cdot e^{\frac{-\omega_{01}\tau}{2Q_1}} \int_0^\infty e^{-\omega_0 v}\left(\cos\big(\widetilde{\omega}_1(\tau+v)\big) - \frac{\sin\big(\widetilde{\omega}_1(\tau+v)\big)}{A_1}\right)\cdot\left(\cos(\widetilde{\omega}_2 v) - \frac{\sin(\widetilde{\omega}_2 v)}{A_2}\right)dv\,.$$

Next step is to evaluate this integral exactly.

Cross-Covariance: Integration Details

After transforming products of trigonometric functions into sums of such functions, the cross-covariance integral representation can be rewritten as a sum

$$K_{12}(\tau) = \frac{K_0}{2} e^{\frac{-\omega_{01}\tau}{2Q_1}} \big(I_1(\tau) + I_2(\tau) - I_3(\tau) - I_4(\tau)\big),$$

where $I_m(\tau)$ are four similar integrals computed below (see the next slide for some explanations).

$$I_1 = \frac{A_1A_2 - 1}{A_1A_2} \int_0^\infty e^{-\omega_0 v} \cos(\widetilde{\omega}_1(\tau + v) + \widetilde{\omega}_2 v)\, dv = \frac{A_1A_2 - 1}{A_1A_2} \cdot \frac{\omega_0 \cos\widetilde{\omega}_1\tau - (\widetilde{\omega}_1 + \widetilde{\omega}_2)\sin\widetilde{\omega}_1\tau}{\omega_0^2 + (\widetilde{\omega}_1 + \widetilde{\omega}_2)^2};$$

$$I_2 = \frac{A_1A_2 - 1}{A_1A_2} \int_0^\infty e^{-\omega_0 v} \cos(\widetilde{\omega}_1(\tau + v) - \widetilde{\omega}_2 v)\, dv = \frac{A_1A_2 + 1}{A_1A_2} \cdot \frac{\omega_0 \cos\widetilde{\omega}_1\tau - (\widetilde{\omega}_1 - \widetilde{\omega}_2)\sin\widetilde{\omega}_1\tau}{\omega_0^2 + (\widetilde{\omega}_1 - \widetilde{\omega}_2)^2};$$

$$I_3 = \left(\frac{1}{A_1} + \frac{1}{A_2}\right) \int_0^\infty e^{-\omega_0 v} \sin(\widetilde{\omega}_1(\tau + v) + \widetilde{\omega}_2 v)\, dv = \frac{A_1 + A_2}{A_1A_2} \cdot \frac{\omega_0 \sin\widetilde{\omega}_1\tau + (\widetilde{\omega}_1 + \widetilde{\omega}_2)\cos\widetilde{\omega}_1\tau}{\omega_0^2 + (\widetilde{\omega}_1 + \widetilde{\omega}_2)^2};$$

$$I_4 = \left(\frac{1}{A_1} - \frac{1}{A_2}\right) \int_0^\infty e^{-\omega_0 v} \sin(\widetilde{\omega}_1(\tau + v) - \widetilde{\omega}_2 v)\, dv = \frac{A_2 - A_1}{A_1A_2} \cdot \frac{\omega_0 \sin\widetilde{\omega}_1\tau + (\widetilde{\omega}_1 - \widetilde{\omega}_2)\cos\widetilde{\omega}_1\tau}{\omega_0^2 + (\widetilde{\omega}_1 - \widetilde{\omega}_2)^2}.$$

Cross-Covariance: a Clarification

The computations on the previous slide made significant use of the relations

$$\int_0^\infty e^{-\operatorname{Re}\zeta\cdot v}\cos(\varphi - v\cdot\operatorname{Im}\zeta)\,dv = \operatorname{Re}\int_0^\infty e^{j\varphi-\zeta v}dv = \frac{\operatorname{Re}\zeta\cdot\cos\varphi + \operatorname{Im}\zeta\cdot\sin\varphi}{|\zeta|^2}$$

$$\int_0^\infty e^{-\operatorname{Re}\zeta\cdot v}\sin(\varphi - v\cdot\operatorname{Im}\zeta)\,dv = \operatorname{Im}\int_0^\infty e^{j\varphi-\zeta v}dv = \frac{\operatorname{Re}\zeta\cdot\sin\varphi - \operatorname{Im}\zeta\cdot\cos\varphi}{|\zeta|^2}$$

which hold for $\varphi \in \mathbb{R}$ and $\operatorname{Re}\zeta > 0$. These are relatively easy to verify in the following fashion:

$$\int_0^\infty e^{j\varphi-\zeta v}dv = e^{j\varphi}\cdot\int_0^\infty e^{-\zeta v}dv = \left.\frac{e^{-\zeta v}}{-\zeta}\right|_0^{+\infty}\cdot e^{j\varphi} = \frac{e^{j\varphi}}{\zeta} = \frac{e^{j\varphi}\cdot\zeta^*}{|\zeta|^2} =$$

$$= \frac{(\cos\varphi + j\sin\varphi)\cdot(\operatorname{Re}\zeta - j\cdot\operatorname{Im}\zeta)}{|\zeta|^2} =$$

$$= \frac{(\operatorname{Re}\zeta\cdot\cos\varphi + \operatorname{Im}\zeta\cdot\sin\varphi) + j\cdot(\operatorname{Re}\zeta\cdot\sin\varphi - \operatorname{Im}\zeta\cdot\cos\varphi)}{|\zeta|^2}.$$

Cross-Covariance: Finalizing

Now we combine the $I_m(\tau)$ integrals to obtain the main result. During this process, some new constants appear:

$$C = \frac{1}{A_1 A_2} \cdot \left(\frac{(A_1 A_2 - 1)\omega_0 - (A_1 + A_2)(\widetilde{\omega}_1 + \widetilde{\omega}_2)}{\omega_0^2 + (\widetilde{\omega}_1 + \widetilde{\omega}_2)^2} + \frac{(A_1 A_2 + 1)\omega_0 - (A_2 - A_1)(\widetilde{\omega}_1 - \widetilde{\omega}_2)}{\omega_0^2 + (\widetilde{\omega}_1 - \widetilde{\omega}_2)^2} \right),$$

$$D = \frac{1}{A_1 A_2} \cdot \left(\frac{(A_1 + A_2)\omega_0 + (A_1 A_2 - 1)(\widetilde{\omega}_1 + \widetilde{\omega}_2)}{\omega_0^2 + (\widetilde{\omega}_1 + \widetilde{\omega}_2)^2} + \frac{(A_1 A_2 + 1)(\widetilde{\omega}_1 - \widetilde{\omega}_2) + (A_2 - A_1)\omega_0}{\omega_0^2 + (\widetilde{\omega}_1 - \widetilde{\omega}_2)^2} \right).$$

They serve as the ‘quadrature coefficients’ which allow to finally write the cross-covariance in the form of

$$K_{12}(\tau) = \frac{K_0}{2} e^{\frac{-\omega_{01}\tau}{2Q_1}} (C \cos \widetilde{\omega}_1 \tau - D \sin \widetilde{\omega}_1 \tau).$$

This expression, although somewhat cumbersome, is explicit (so it can be numerically evaluated almost instantly on any relatively modern computing device), general (because no additional assumptions have been made) and exact (because we also never made any approximations throughout the whole derivation).

The Cross-Correlation Coefficient

The cross-correlation coefficient is the normalized version of the cross-covariance and is given by the formula $R_{12}(\tau) = \frac{K_{12}(\tau)}{\sigma_1\sigma_2} = \frac{K_{12}(\tau)}{\sqrt{K_1(0)K_2(0)}}$, where the standard deviations σ_i are computed using the auto-covariance $K_i(\tau) = \langle\eta_i(t+\tau)\eta_i(t)\rangle$ as $\sigma_i = \sqrt{K_i(0)}$.

Formally, $K_1(\tau)$ is equivalent to $K_{12}(\tau)$ if we replace all the parameters of the second circuit by those of the first one. This greatly simplifies some of the relations above: for example, the coefficient C becomes $C = \frac{2A_1^2\omega_0^2+4\widetilde{\omega}_1\left(\left(A_1^2+1\right)\widetilde{\omega}_1-A_1\omega_0\right)}{A_1^2\omega_0\left(\omega_0^2+(2\widetilde{\omega}_1)^2\right)} = \frac{Q_1}{\omega_{01}}$. Therefore, we have $K_1(0) = \frac{K_0Q_1}{2\omega_{01}}$ and, by symmetry, $K_2(0) = \frac{K_0Q_2}{2\omega_{02}}$, so that $\frac{K_0}{2\sqrt{K_1(0)K_2(0)}} = \sqrt{\frac{\omega_{01}\omega_{02}}{Q_1Q_2}}$ and

$$R_{12}(\tau) = \sqrt{\frac{\omega_{01}\omega_{02}}{Q_1Q_2}}\, e^{\frac{-\omega_{01}\tau}{2Q_1}}(C\cos\widetilde{\omega}_1\tau - D\sin\widetilde{\omega}_1\tau).$$

This last formula concludes our solution.

Thanks for your attention