



# Применение обобщённых функций для решения задач апертурной теории во временной области

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# Основы теории обобщённых функций

$$\langle f, \varphi \rangle = \langle f(\mathbf{x}), \varphi(\mathbf{x}) \rangle := \mathcal{D}f(\mathbf{x})\varphi(\mathbf{x})|_{\mathbb{R}^n}$$

$$\langle f(A\mathbf{x} + \vec{b}), \varphi(\mathbf{x}) \rangle = \frac{1}{|\det A|} \langle f(\mathbf{x}), \varphi(A^{-1}(\mathbf{x} - \vec{b})) \rangle, \quad \langle f(ay + b), \varphi(y) \rangle = \frac{1}{|a|} \langle f(x), \varphi\left(\frac{x - b}{a}\right) \rangle$$

$$\langle \frac{\partial^{i_1+\dots+i_n} f}{\partial r_1^{i_1} \dots \partial r_n^{i_n}}, \varphi \rangle = (-1)^{i_1+\dots+i_n} \langle f, \frac{\partial^{i_1+\dots+i_n} \varphi}{\partial r_1^{i_1} \dots \partial r_n^{i_n}} \rangle, \quad \langle f', \varphi \rangle = -\langle f, \varphi' \rangle, \quad \langle f^{(m)}, \varphi \rangle = (-1)^m \langle f, \varphi^{(m)} \rangle$$

$$\langle g(\mathbf{y})f(\mathbf{x}), \varphi(\mathbf{x}, \mathbf{y}) \rangle = \langle f(\mathbf{x})g(\mathbf{y}), \varphi(\mathbf{x}, \mathbf{y}) \rangle = \langle g(\mathbf{y}), \langle f(\mathbf{x}), \varphi(\mathbf{x}, \mathbf{y}) \rangle \rangle$$

$$\langle \delta(x), \varphi(x) \rangle = \varphi(0), \delta(\mathbf{x}) := \delta(r_1) \cdot \dots \cdot \delta(r_n)$$

$$\langle \mu(\mathbf{x})\delta_S, \varphi(\mathbf{x}) \rangle = \mathcal{D}\mu(\mathbf{x})\varphi(\mathbf{x})|_S$$

# Импульсная характеристика апертуры

$$E(\mathbf{h}, t) = \mathcal{D} \int_{S_a} dE(\mathbf{r}_a, \mathbf{h}, t) = \mathcal{D} \frac{q(\mathbf{r}_a) \cdot s \left( t - \frac{|\mathbf{h} - \mathbf{r}_a|}{c} \right)}{|\mathbf{h} - \mathbf{r}_a|} dS_{\mathbf{r}_a}$$

$$s(t) = (s * \delta)(t) := \mathcal{D} \int_{-\infty}^{\infty} s(r) \delta(t - r) dr$$

$$E(\mathbf{h}, t) = \mathcal{D} \int_{-\infty}^{\infty} s(r) dr \mathcal{D} \frac{q(\mathbf{r}_a)}{|\mathbf{h} - \mathbf{r}_a|} \cdot \delta \left( t - \frac{|\mathbf{h} - \mathbf{r}_a|}{c} - r \right) dS_{\mathbf{r}_a} = (s * E_{\delta})(t)$$

$$E_{\delta}(\mathbf{r}_a, t) = \mathcal{D} \frac{q(\mathbf{r}_a)}{|\mathbf{h} - \mathbf{r}_a|} \cdot \delta \left( t - \frac{|\mathbf{h} - \mathbf{r}_a|}{c} \right) dS_{\mathbf{r}_a}$$

# Элемент объёма многообразия

$$S = \{ \mathbf{h} \in \mathbb{R}^n \mid \mathbf{h} = \mathbf{h}(\vec{u}), \vec{u} \in U \subseteq \mathbb{R}^k \}$$

$$\mathbf{g}(\vec{u}) = \left[ \begin{matrix} \mathbf{g}_m & (\vec{u}) \\ p & k \end{matrix} \right]_{p \leq k}^{m \leq k}, \quad \mathbf{g}_m & (\vec{u}) = \bigcup_{i=1}^n \frac{\partial r_i(\vec{u})}{\partial u_m} \frac{\partial r_i(\vec{u})}{\partial u_p} = \left( \frac{\partial \mathbf{h}(\vec{u})}{\partial u_m}, \frac{\partial \mathbf{h}(\vec{u})}{\partial u_p} \right)$$

$$\int_S \varphi(\mathbf{h}) dS = \int_U \varphi(\mathbf{h}(\vec{u})) \cdot \sqrt{|\det \mathbf{g}|} \cdot |d^k \vec{u}| = \int_U \varphi(\mathbf{h}(\vec{u})) \cdot \sqrt{(\mathbf{D}\mathbf{h}(\vec{u}))^\top (\mathbf{D}\mathbf{h}(\vec{u}))} \cdot |d^k \vec{u}|$$

$$k = 1, \mathbf{h} = \mathbf{h}(t) \Rightarrow dL = \sqrt{\det(\mathbf{r}\mathbf{h}'(t)^\top \cdot \mathbf{r}\mathbf{h}'(t))} dt = \sqrt{(\mathbf{h}'(t), \mathbf{h}'(t))} = |\mathbf{r}\mathbf{h}'(t)| dt$$

$$k = 2, n = 3, \mathbf{h} = \mathbf{h}(u, v) \Rightarrow dS = |[r_u \mathbf{h}', r_v \mathbf{h}']| du dv$$

$$k = n - 1, S = \{ \mathbf{h} \in \mathbb{R}^n \mid F(\mathbf{h}) = 0 \} \Rightarrow dS = \frac{|\vec{\nabla} F(\mathbf{h}(\vec{u}))|}{|F'_{r_1}(\mathbf{h}(\vec{u}))|} |d^{n-1} \vec{u}|$$

**Связь дельта-функции  
нелинейного n-мерного выражения  
с простым слоем на поверхности,  
где оно обращается в ноль**

$$\delta(f(x)) = \underset{f(x_0)=0}{\star} \frac{\delta(x - x_0)}{|f'(x_0)|}$$

$$\langle \delta(F(\vec{r})), \varphi(\vec{r}) \rangle = \int_{\mathbb{R}^{n-1}}^{\infty} |d^{n-1}\vec{u}| \delta(F(r_1, \vec{u})) \varphi(r_1, \vec{u}) dr_1$$

$$\delta(F(\vec{r})) = \frac{1}{|\vec{\nabla}_{\vec{r}} F(\vec{r})|} \cdot \delta_{\{\vec{r} \in \mathbb{R}^n \mid F(\vec{r}) = 0\}}$$

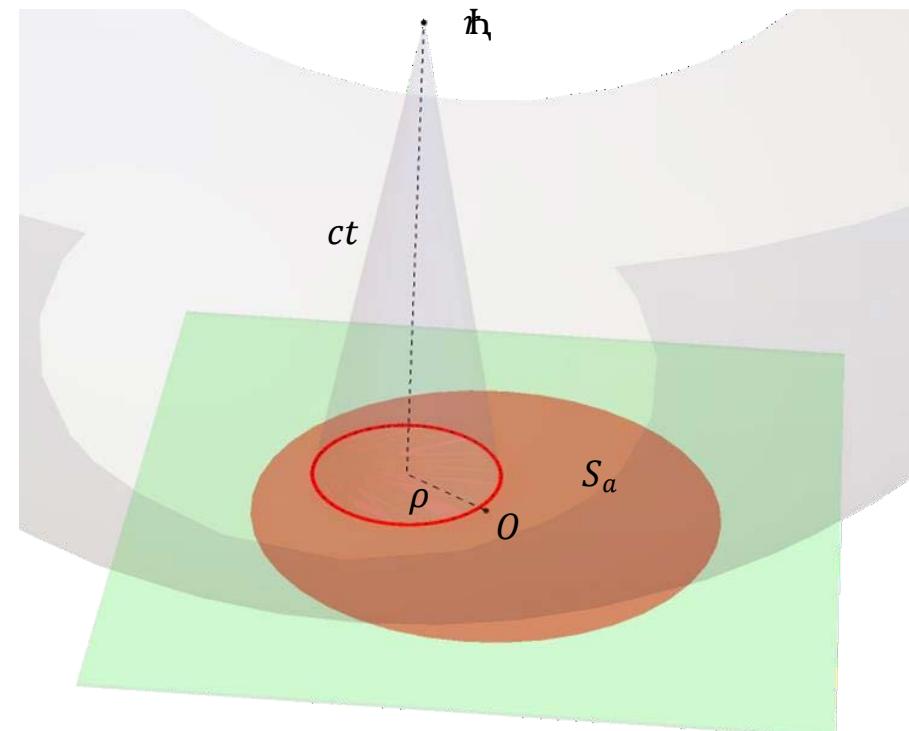
# Случай плоской апертуры

$$E_\delta(\hbar t) = \underset{U}{\mathcal{D}} \frac{q(u, v, 0)}{R} \cdot \delta\left(t - \frac{R}{c}\right) du dv, \quad R = \sqrt{(x - u)^2 + (y - v)^2 + z^2}$$

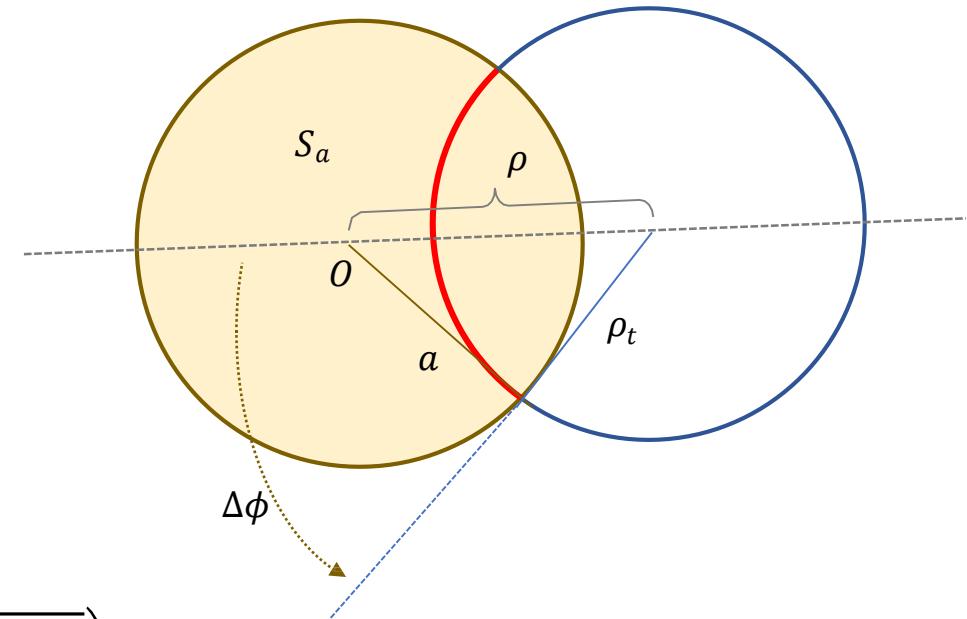
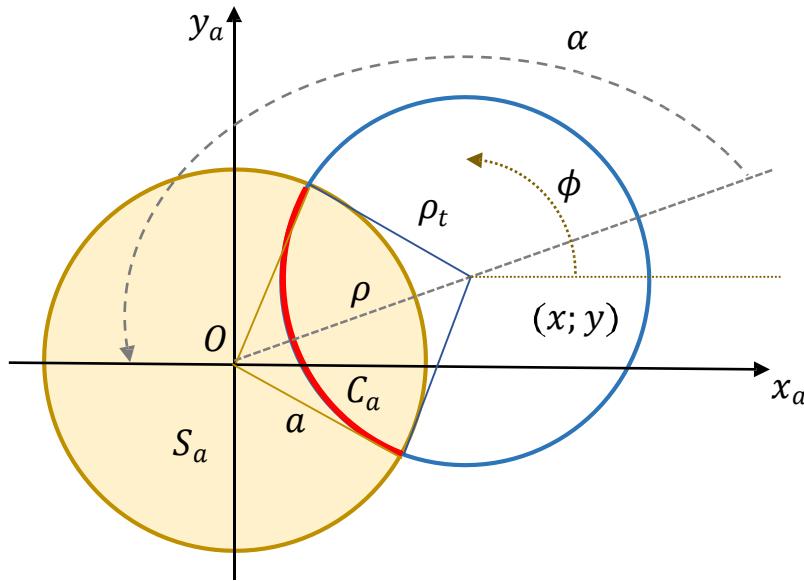
$$t - \frac{R}{c} = 0 \Rightarrow (u; v) \in C_a: \begin{cases} u = x + \rho_t \cdot \cos \phi, \\ v = y + \rho_t \cdot \sin \phi, \end{cases} \quad \rho_t := \sqrt{(ct)^2 - z^2}, \quad \phi \in \Phi \subseteq [0; 2\pi]$$

$$\delta\left(t - \frac{R}{c}\right) = \frac{\delta_{C_a}}{\left| \vec{\nabla}_{(u;v)} \left( t - \frac{R}{c} \right) \right|} = \frac{c \cdot R \cdot \delta_{C_a}}{\sqrt{(x - u)^2 + (y - v)^2}} = \frac{c R \delta_{C_a}}{\rho_t}$$

$$dC_a = \rho_t d\phi \Rightarrow E_\delta(\hbar t) = c \cdot \underset{\Phi}{\mathcal{D}} q(x + \rho_t \cdot \cos \phi, y + \rho_t \cdot \sin \phi, 0) \cdot d\phi$$



# Круглая апертура

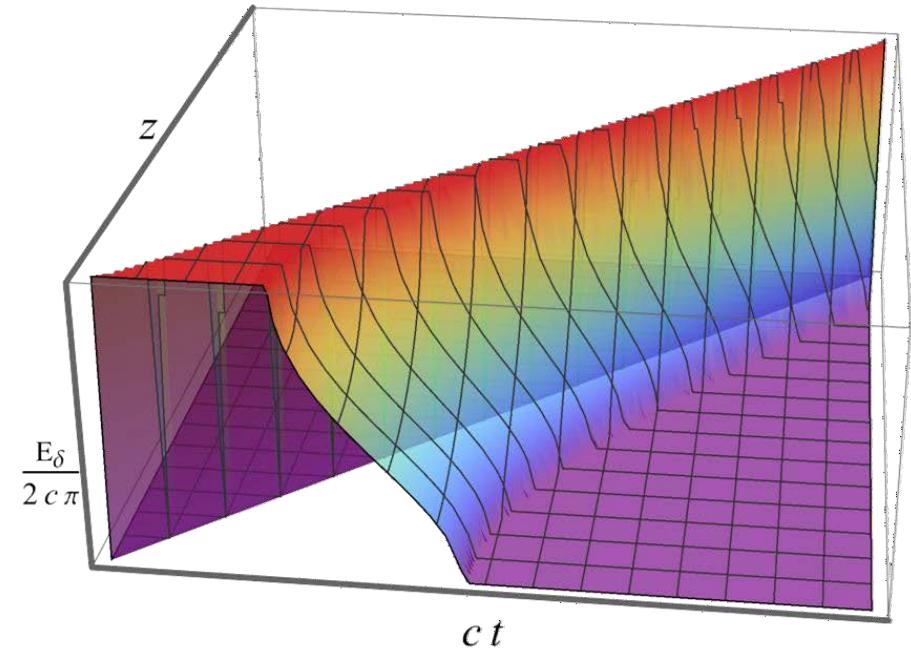
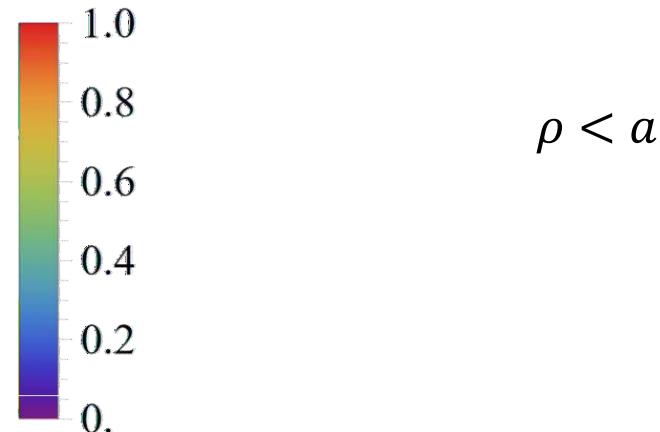
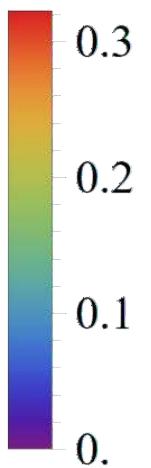
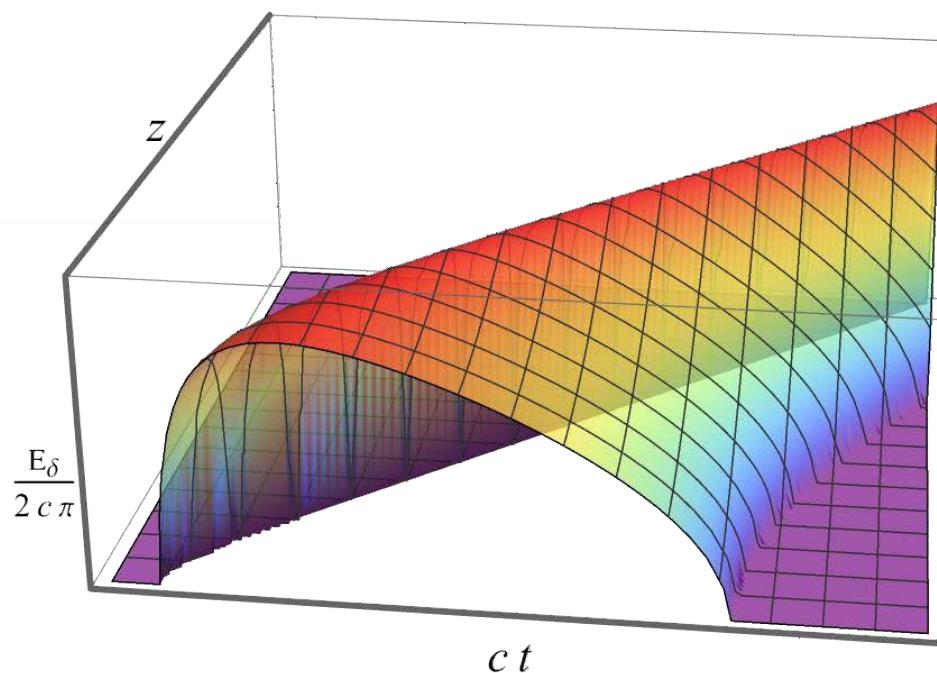


$$E_\delta(\eta t) = \begin{cases} 2\pi c \cdot \Theta(a - \rho), & t \in \left( \frac{z}{c}; \frac{\sqrt{z^2 + (a - \rho)^2}}{c} \right), \\ 2c \cdot \Delta\phi, & t \in \left( \frac{\sqrt{z^2 + (\rho - a)^2}}{c}; \frac{\sqrt{z^2 + (\rho + a)^2}}{c} \right), \\ 0, & t \in \left( 0; \frac{z}{c} \right) \cup \left( \frac{\sqrt{z^2 + (\rho + a)^2}}{c}; +\infty \right), \end{cases}$$

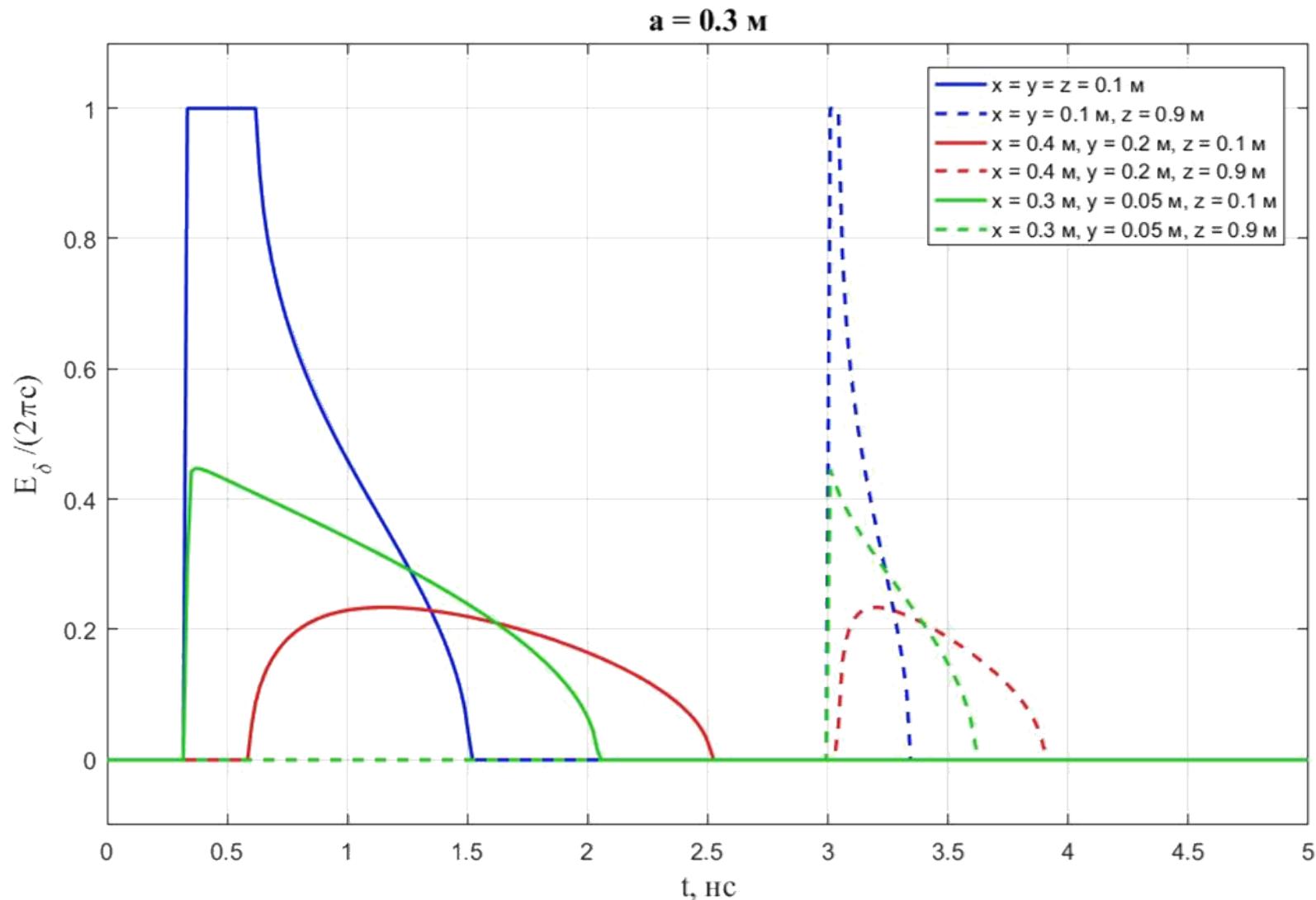
$\rho = \sqrt{x^2 + y^2}$

$$\Delta\phi = \arccos \frac{\rho^2 + \rho_t^2 - a^2}{2\rho\rho_t}$$

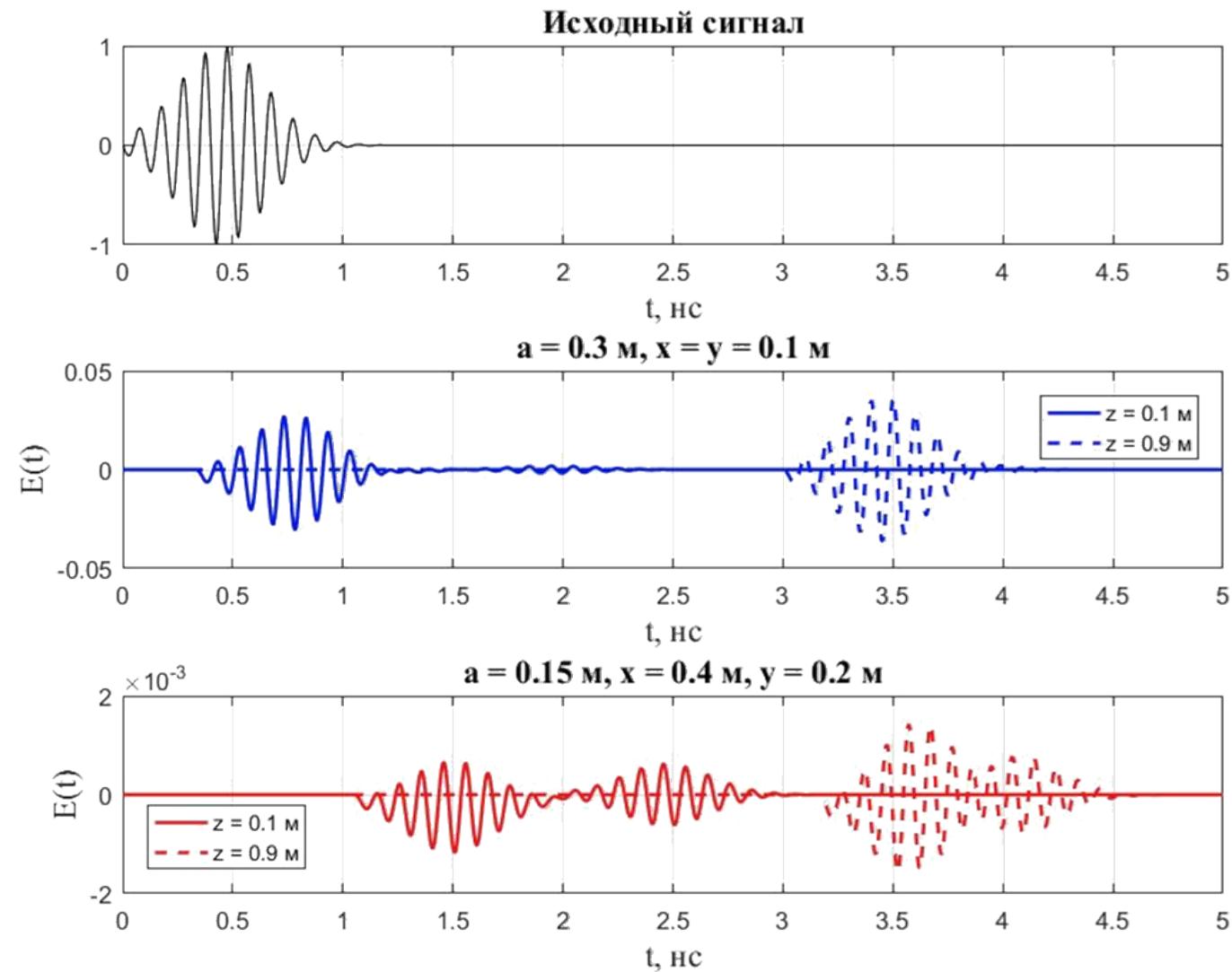
# Круглая апертура

 $\rho > a$  $\rho < a$ 

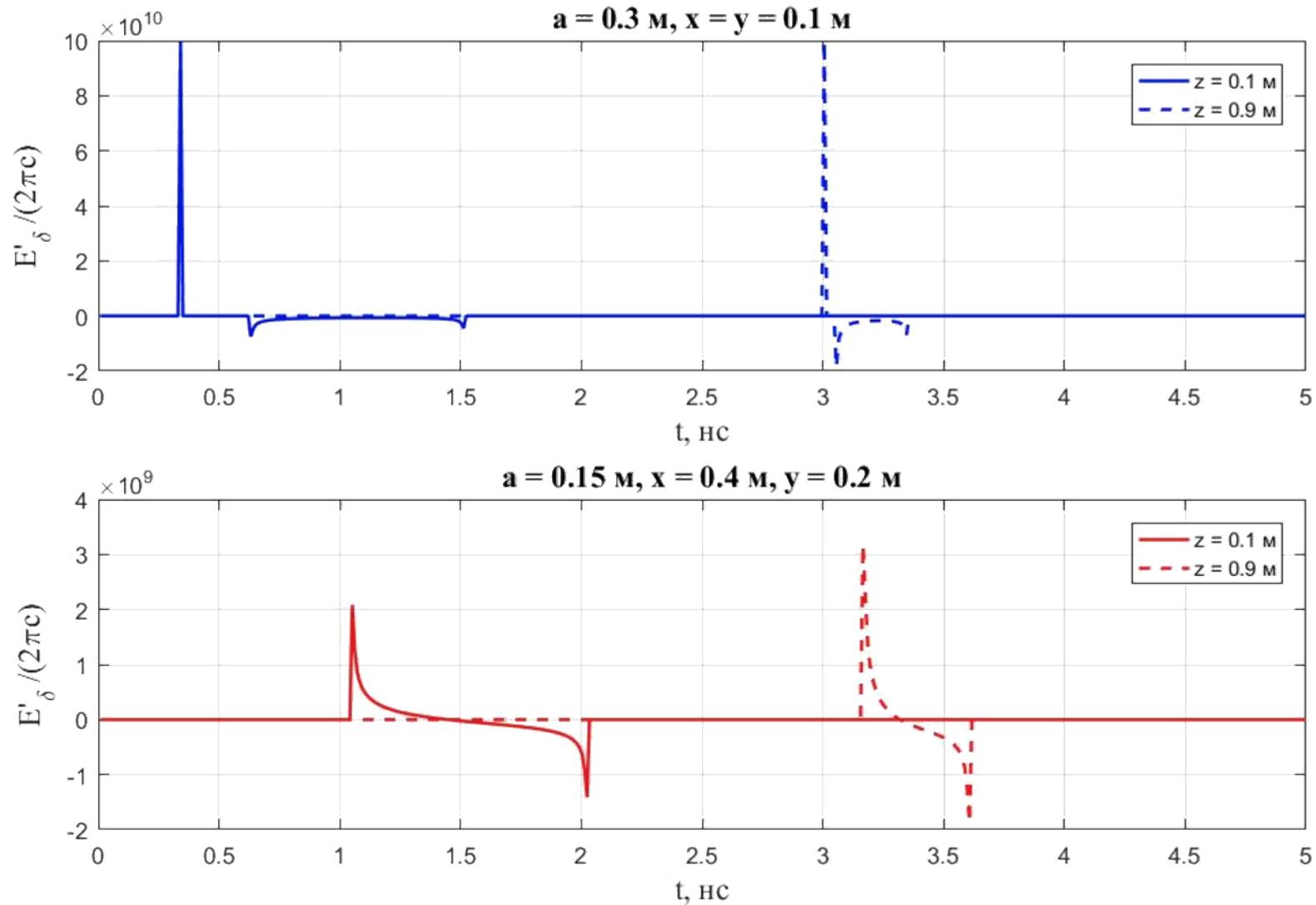
# Круглая апертура



# Круглая апертура



# Круглая апертура



# Сферическая апертура

$$\begin{aligned}x_a &= a \sin \theta_a \cos \phi_a; \\y_a &= a \sin \theta_a \sin \phi_a; \\z_a &= a \cos \theta_a;\end{aligned}$$

$$\begin{aligned}dS &= a^2 \sin \theta_a d\theta_a d\phi_a \\ \mathbf{h} &= (0; 0; r)\end{aligned}$$

$$\begin{aligned}\theta_a^{(0)}(t) &= \arccos \frac{a^2 + r^2 - (ct)^2}{2ra} \\ \theta_a^{(max)} &= \arccos \frac{a}{r}\end{aligned}$$

$$\begin{aligned}E_\delta(r, t) &= \int_0^{2\pi} d\phi_a \int_0^{\theta_a^{(max)}} \frac{q(\mathbf{h}_a)}{|\mathbf{h} - \mathbf{h}_a|} \cdot \delta\left(t - \frac{|\mathbf{h} - \mathbf{h}_a|}{c}\right) a^2 \sin \theta_a d\theta_a = \\ &= \Theta\left(t - \frac{r - a}{c}\right) \cdot \Theta\left(\frac{\sqrt{r^2 - a^2}}{c} - t\right) \frac{ca}{r} \int_0^{2\pi} q\left(\mathbf{h}_a(\theta_a^{(0)}(t), \phi_a)\right) d\phi_a\end{aligned}$$

$$\begin{aligned}E(\mathbf{h}, t) &= \int_0^{2\pi} d\phi_a \int_0^{\theta_a^{(max)}} \frac{q(\mathbf{h}_a(\theta_a, \phi_a))}{|\mathbf{h} - \mathbf{h}_a|} \cdot s\left(t - \frac{|\mathbf{h} - \mathbf{h}_a|}{c}\right) a^2 \sin \theta_a d\theta_a = \\ &= \int_{\frac{r-a}{c}}^{\frac{\sqrt{r^2-a^2}}{c}} s(t-r) \int_0^{2\pi} \frac{ca}{r} q\left(\mathbf{h}_a\left(\theta_a^{(0)}(r), \phi_a\right)\right) d\phi_a dr = \int_{-\infty}^{\infty} s(t-r) E_\delta(r, r) dr\end{aligned}$$

# Цилиндрическая апертура

$$\begin{aligned} x_a &= a \cos \phi_a \\ ; \quad \Phi_a &= a \sin \phi_a \\ ; \quad z_a &= z_a; \end{aligned}$$

$$dS = a \cdot d\phi_a dz_a$$

$$dC_a = \sqrt{1 + \left(\frac{\partial z_a}{\partial \phi_a}\right)^2} d\phi_a = \frac{\sqrt{z_a^2 + (ra \cos \phi_a)^2}}{|z_a|} d\phi_a$$

$$E_\delta(r, t) \equiv \mathcal{D} \int_{-\infty}^{\infty} \int_0^\pi \frac{a}{|\vec{h} - \vec{h}_a|} \cdot \frac{1}{\left| \vec{\nabla}_{(\phi_a; z)} \left( t - \frac{1}{c} \sqrt{a^2 + r^2 + z_a^2 - 2ra \sin \phi_a} \right) \right|} \cdot dC_a d\phi_a$$

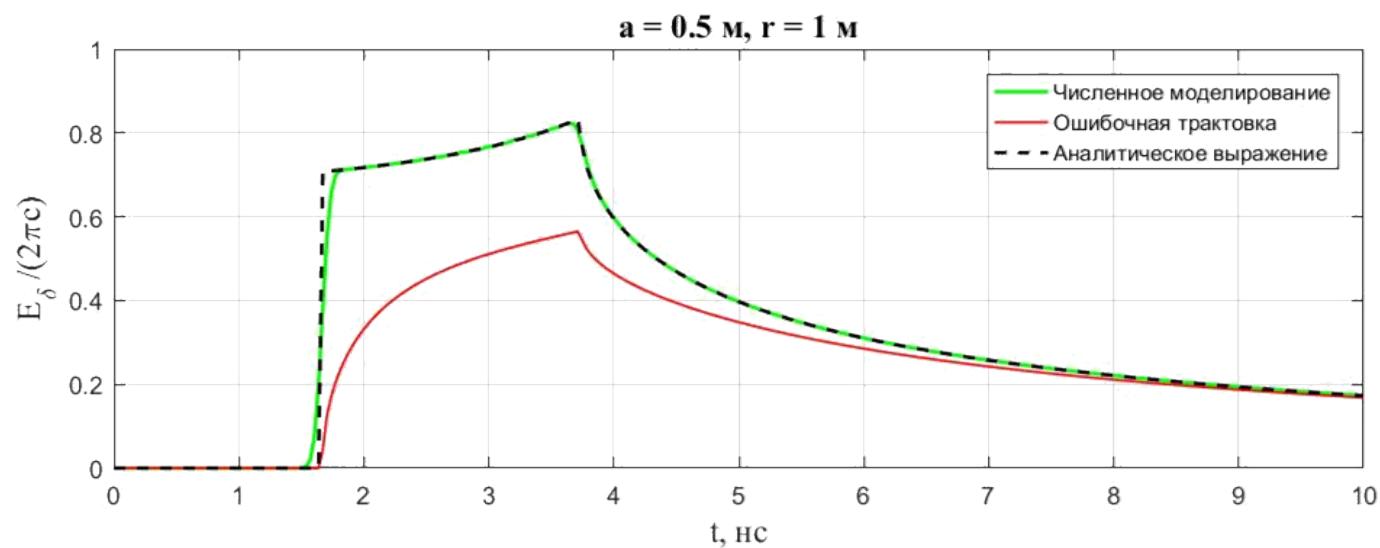
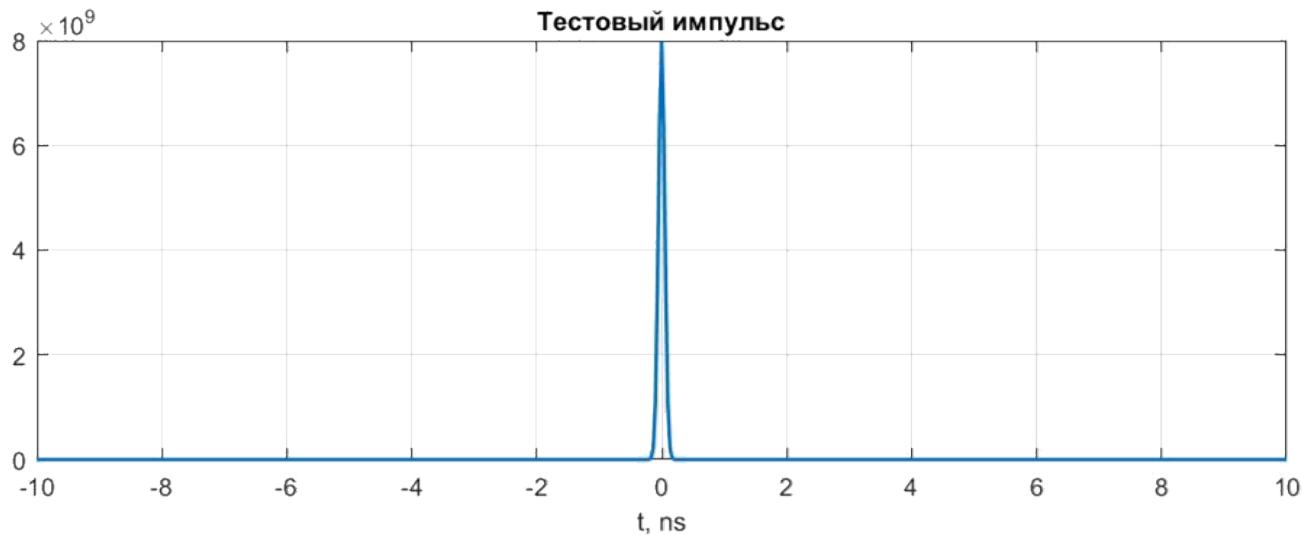
$$E_\delta(r, t) = \mathcal{D} \frac{2ac}{\sqrt{(ct)^2 - a^2 - r^2 + 2ra \sin \phi_a}} \cdot d\phi_a = \begin{cases} \frac{8ac \cdot \mathcal{F}(\Delta\phi_a(t), \kappa)}{\sqrt{(ct)^2 - (a - r)^2}}, & t \in \left(\frac{r - a}{c}; \frac{\sqrt{a^2 + r^2}}{c}\right) \\ \frac{8ac \cdot \mathcal{F}\left(\frac{\pi}{2}, \kappa\right)}{\sqrt{(ct)^2 - (a - r)^2}}, & t > \frac{\sqrt{a^2 + r^2}}{c}. \end{cases}$$

$$\Phi = \left[ \frac{\pi}{2} - \Delta\phi_a(t); \frac{\pi}{2} + \Delta\phi_a(t) \right] \cap \left[ \frac{\pi}{2}; \frac{\pi}{2} \right]$$

$$\Delta\phi_a(t) = \arccos \frac{a^2 + r^2 - (ct)^2}{2ra}$$

$$\mathcal{F}(\alpha, \kappa) = \mathcal{D} \int_0^\alpha \frac{d\theta}{\sqrt{1 - \kappa^2 \sin^2 \theta}}$$

# Цилиндрическая апертура



**Спасибо за внимание**