Framed link as a topological invariant for polar flows

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Abstract
A classification problem for polar flows directly follows from classical results where topological classification of gradient-like flows were reduced to combinatorial description of mutual arrangement of nodal equilibria and invariant manifolds of saddles. However, in case of four-dimensional ambient manifold a non-wandering set can contain saddles with wildly embedded closures of separatrices. This makes classification problem impossible to solve in combinatorial terms for higher dimensions. However, if a polar flow, defined on a four-dimensional manifold, has saddles with only two-dimensional invariant manifolds then heteroclinic intersections are empty. So, the intersection of two-dimensional invariant manifolds with some smooth three-dimensional sphere is a link. Here is the construction of a Kirby diagram for a polar flow consisting of the smoothly embedded three-dimensional sphere, the link and its framing, defined by the flow. Eventually, we prove that a Kirby diagram is a complete topological invariant for polar flows on four-dimensional manifolds.

Introduction

A flow f^t on smooth connected closed orientable manifold M^n of dimension n is called *a gradient-like flow* if:

- its non-wandering set Ω_{f^t} consists of finite number of hyperbolic states of equilibrium;
- stable and unstable manifolds of saddles have only transversal intersection.

A gradient-like flow $f^t: M^n \to M^n$ is called a *polar flow* if its non-wandering set consists of two nodal equilibria and a finite number of saddle equilibria. We will say that a state of equilibrium has a type $(i, n-i)$ if its unstable manifold has a dimension $i \in \{0, \dots, n\}$.

In [1], [2] E. A. Leontovich and A. G. Mayer had presented an invariant for gradient-like flows on the sphere $\mathbb{S}^2$ called a *Leontovich-Mayer scheme*. Later M. Peixoto generalized this result. In [3] he got a complete invariant for flows defined on an arbitrary surface M^2 in the form of a *framed Peixoto graph*.

Ya. L. Umanskiy presented a complete invariant for gradient-like flows on three-dimensional manifolds in [4]. This invariant is similar to Leontovich-Mayer scheme. It contains a combinatorial description of boundaries of cells of a flow and of contiguity of cells to nodal equilibrium states.

In [5] a complete topological classification was obtained for gradient-like flows without heteroclinic intersections, defined on manifolds of dimension $n \geq 3$, such that the set of saddle equilibria consists of saddles having a type $(1, n-1)$ or $(n-1, 1)$. Exactly, V. Z. Grines and E. Ya. Gurevich constructed a complete invariant called the *bi-colored graph*. In the paper [6] a realization problem for such flows has been solved.

In general, gradient-like flows can have saddles of types different to $(1, n-1)$ and $(n-1, 1)$. Closures of invariant manifolds of these saddles can be wildly embedded at nodal points. An example of a gradient-like flow with the one wildly embedded separatrix of the saddle equilibrium was constructed in [7]. Thus, a combinatorial classification is impossible for gradient-like flows on manifolds of dimension $n > 3$. Nevertheless, in case of $n = 4$ E. V. Zhuzhoma and V. S. Medvedev obtained a topological classification of gradient-like flows with three states of equilibrium. It was proven in [8] that the class of topological equivalence of such flows is unique and closures of invariant manifold of saddle are locally flat at every point.

Here we consider a class $\mathcal{P}(M^4)$ of polar flows such that for any $f^t \in \mathcal{P}(M^4)$ all its saddles have a type $(2, 2)$.

Kirby diagram for a polar flow

Let $f^t \in \mathcal{P}(M^4)$. It follows from [9, Proposition 3.2.] that the ambient manifold M^4 of the flow f^t is simply connected. From [11], [10] it follows that there exists an energy function $\varphi: M^4 \to [0, 4]$ for the f^t such that it is a self-indexing Morse function strictly decreasing along unclosed trajectories; the set $\mathrm{Cr}(\varphi)$ of critical points coincides with the set Ω_{f^t} and $\varphi(p) = \dim W^u_p$ for any $p \in \mathrm{Cr}(\varphi)$, where the W^u_p is an unstable manifold of the p. Set $\Sigma^3_c = \varphi^{-1}(c)$, $M^4_c = \varphi^{-1}([0, c])$. For any $c \in (0, 2) \cup (2, 4)$ the manifold Σ^3_c is a smoothly embedded in M^4 sphere. Let us denote by α, ω the source and sink equilibria of the flow f^t, respectively, by $\Omega^2_{f^t}$ the set of saddle equilibria. From definition it follows that for any $p, q \in \Omega^2_{f^t}$ $W^u_p \cap W^s_q = \emptyset$ and then $\mathrm{cl}\, W^u_p = W^u_p \cup \{\omega\}$, $\mathrm{cl}\, W^s_q = W^s_q \cup \{\alpha\}$, where the W^s_q is a stable manifold of the q. For $c_1 \in (0, 2)$, $c_2 \in (2, 4)$ put $l_{p,c_1} = \Sigma^3_{c_1} capW^u_p$, $l_{p,c_2} = \Sigma^3_{c_2} \cap W^s_p$. For any $p \in \Omega^2_{f^t}$ the sets l_{p,c_1}, l_{p,c_2} are knots on the spheres $\Sigma^3_{c_1}$, Σ_{c_2}, respectively. Let us denote $L_{p,c_1} = \bigcup\limits_{p \in \Omega^2_{f^t}} l_{p,c_1}$, $L_{p,c_2} = \bigcup\limits_{p \in \Omega^2_{f^t}} l_{p,c_2}$.

Here and everywhere below, for the flow $f'^t \in \mathcal{P}(M^4)$ we will denote by $\Sigma'^3_{c_1}$, L'_{c_1} etc. objects having the same meaning as $\Sigma^3_{c_1}$, L_{c_1} etc. for the flow f^t. From definition of a topological equivalence the following proposition is true.

The link L_{c_1} is a topological invariant for polar flows. To obtain a complete invariant we need equip the link L_{c_1} by some information. Set $N_{c_2} \subset \Sigma^3_{c_2}$ is a disjoint union of solid tori that are closed tubular neighbourhoods of knots forming the link L_{c_2}. Put $\Pi_{p,c_2} \subset \Pi_{p,c_2}$ is a tubular neighbourhood of the knot l_{p,c_1} and μ_{p,c_2} is its canonical meridian. The flow's f^t trajectories define a diffeomorphism $\eta_{c_2c_1}: \Sigma^3_{c_2} \setminus L_{c_2} \to \Sigma^3_{c_1} \setminus L_{c_1}$ by the rule $\eta_{c_2c_1}(x) = \mathcal{O}_{f^t}(x) \cap \Sigma^3_{c_1}$, where $\mathcal{O}_{f^t}(x)$ is a trajectory of the flow f^t passing through the $x \in \Sigma^3_{c_2} \setminus L_{c_2}$. The knot $\tilde{l}_{p,c_1} = \eta_{c_2c_1}(\mu_{p,c_2})$ we will call a *framing* of the knot l_{p,c_1}. The set $\{l_{p,c_1}, \tilde{l}_{p,c_1}\}$ we will call a *Kirby diagram* for the flow f^t.

Note, that for any $c, \hat{c}$ belonging to the same connectivity component of the set $(0, 2) \cup (2, 4)$ the spheres Σ^3_c and $\Sigma^3_{\hat{c}}$ are diffeomorphic by the diffeomorphism $\eta_{c\hat{c}}: \Sigma^3_c \to \Sigma^3_{\hat{c}}$ defined by the rule $\eta_{c\hat{c}}(x) = \mathcal{O}_{f^t}(x) \cap \Sigma^3_{\hat{c}}$ for any $x \in \Sigma^3_c$. So, the definition of a Kirby diagram and the following theorem for a flow does not depend on choice of the c_1, c_2.

Theorem 1. *The flows $f^t, f'^t \in \mathcal{P}(M^4)$ are topologically equivalent if and only if there exists the homeomorphism $h: \Sigma^3_{c_1} \to \Sigma'^3_{c_1}$ such that:*

1. $h(L_{c_1}) = L'_{c_1}$;

2. $h(\tilde{l}_{p,c_1}) = \tilde{l}'_{p',c_1}$ for any couple $l_{p,c_1} \in \Sigma^3_{c_1}$, $l'_{p',c_1} = h(l_{p,c_1})$.

The necessity of the theorem's conditions follows from definition of topological equivalence and Morse function's properties. To prove sufficiency we construct the piecewise defined homeomorphism $G: M^4 \to M^4$ such that $f^t = G^{-1} f'^t G$.

To start with, the homeomorphisms $G_p: V_p \to V_{p'}$ are defined for all $p \in \Omega^2_{f^t}$, where $V_p, V_{p'}$ are compact canonical neighbourhoods of the saddles p, p', respectively. The restriction of mapping $\eta_{c_2c_1}$ on V_p induces the homeomorphism $H_{c_1}: \Sigma^3_{c_1} \to \Sigma'^3_{c_1}$ that can be continued to the homeomorphisms $G_{c_1}: M^4_{c_1} \to M'^4_{c_1}$ and $\psi_p: L^u_p \to L^u_{p'}$ by trajectories' segments. The mappings G_{c_1} and ψ_p coincide with the H_{c_1} on the set $\Sigma^3_{c_1}$ and with the G_p on the Π_u, respectively. Using the gluing lemma we have the homeomorphism $G_+: M^4_{c_1} \cup \bigcup\limits_{p \in \Omega^2_{f^t}} L^u_p \cup \bigcup\limits_{p \in \Omega^2_{f^t}} V_p \to M'^4_{c_1} \cup \bigcup\limits_{p' \in \Omega^2_{f'^t}} L^u_{p'} \cup \bigcup\limits_{p' \in \Omega^2_{f'^t}} V_{p'}$. At last, it is possible to continue the G_+ to the homeomorphism $G: M^4 \to M'^4$ by trajectories segments applying the gluing lemma.

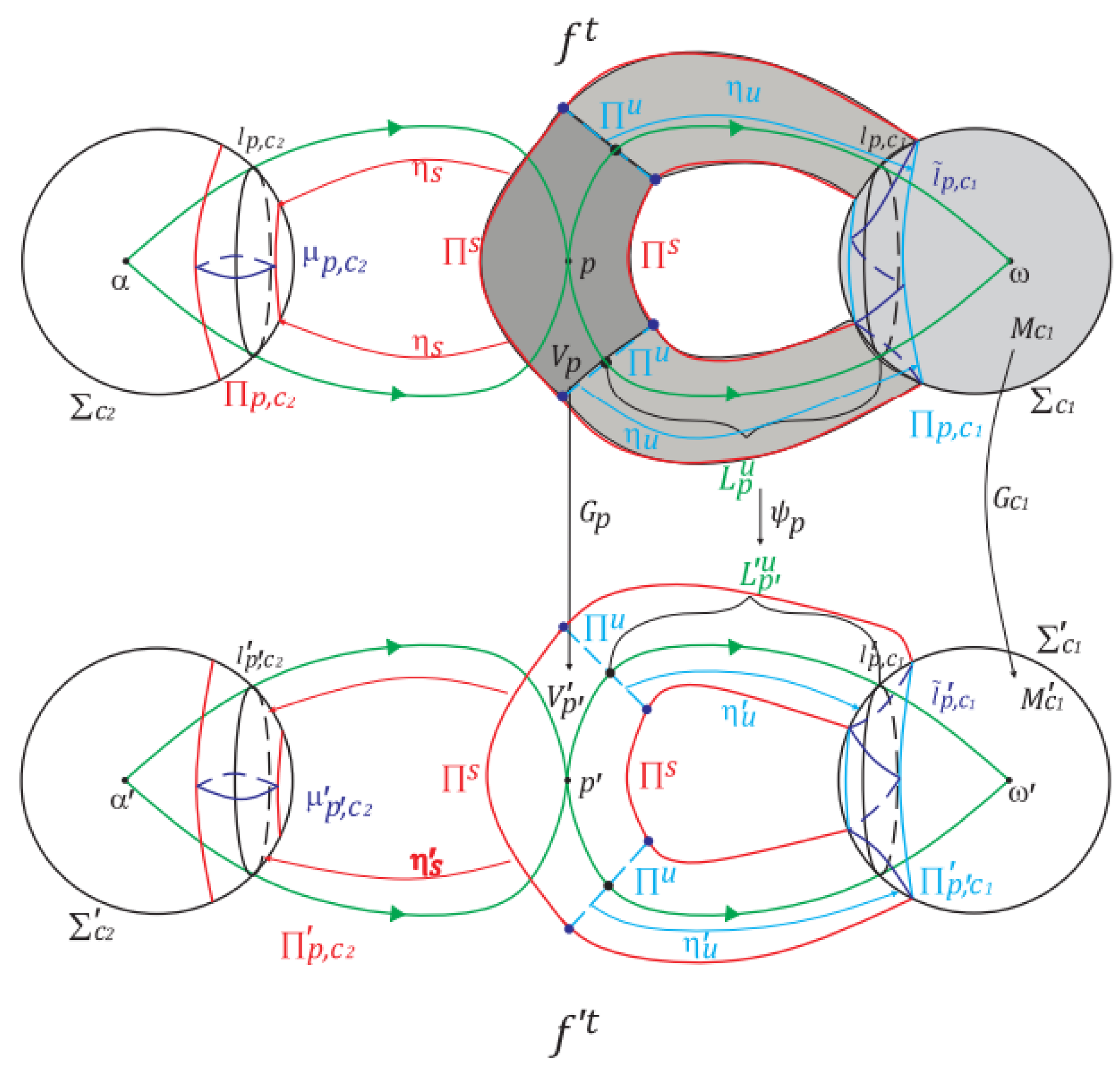

Figure 1: Construction of the homeomorphism of the manifolds carrying the flows f^t, f'^t.

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