On Gradient-like Flows with One Saddle Equilibrium of Type (2,2)

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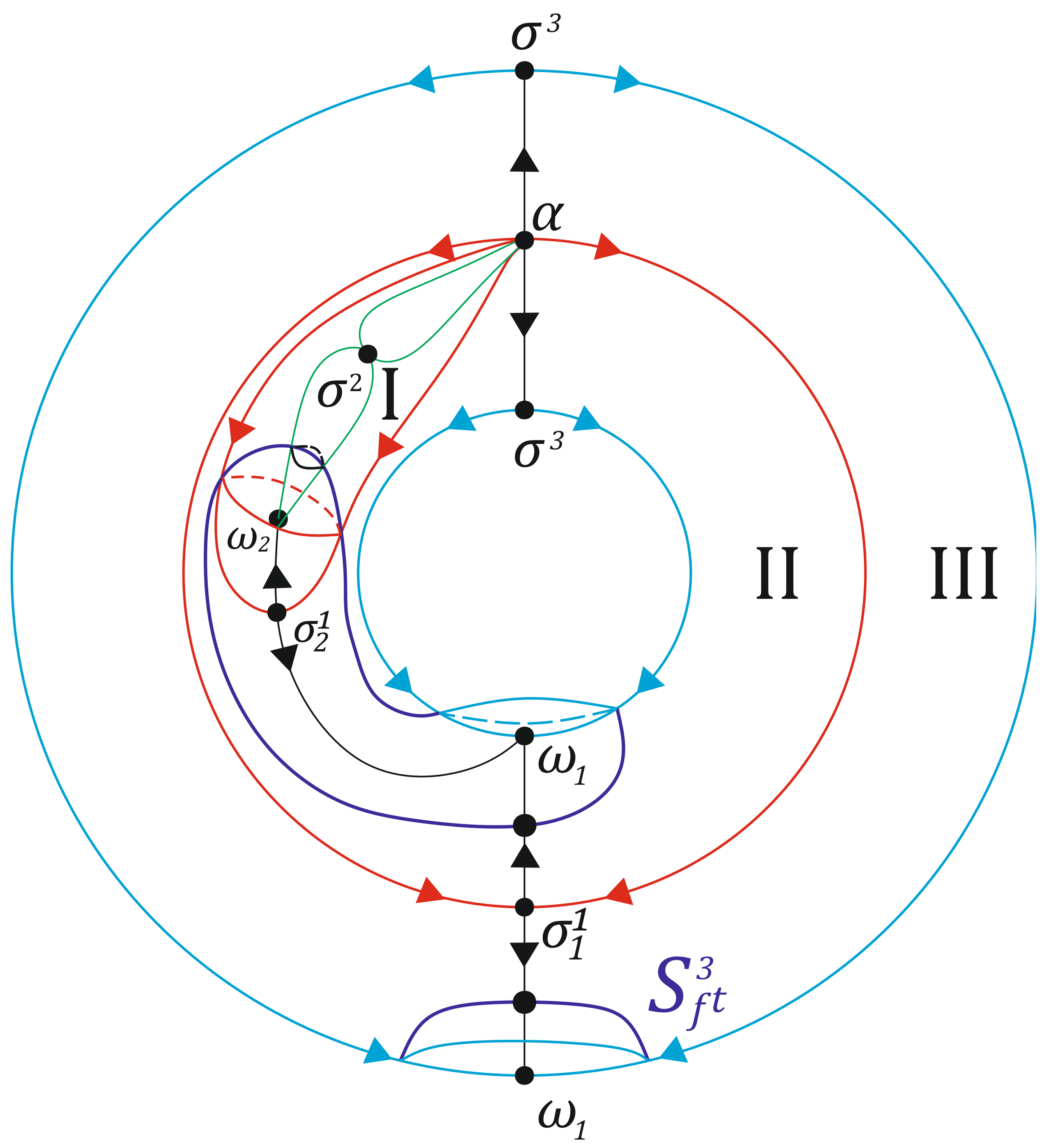

Figure 1: Phase portrait of $f^t \in G_{1,1}(M^4)$

Let f^t be a gradient-like flow on a closed manifold M^4 of dimension four. We say that an equilibrium $p \in \Omega_{f^t}$ is of *type* $(i, 4-i)$ if the dimension of its unstable manifold W^u_p equals $i \in \{0, \dots, 4\}$. Denote by k_i the number of equilibria of type $(i, 4-i)$.

Theorem 1. *Suppose that invariant manifolds of different saddle equilibria of f^t do not intersect each other. Then the number* $g_{f^t} = \dfrac{k_1 + k_3 - k_0 - k_4 + 2}{2}$ *is integer non-negative. Moreover, the manifold M^4 is homeomorphic to the connected sum of the complex projective plane $\mathbb{C}\mathrm{P}^2$ and g_{f^t} copies of the direct product $\mathbb{S}^3 \times \mathbb{S}^1$ if and only if $k_2 = 1$.*

The statement "if" follows from [1, Theorem 1].

Let $G_{g,1}(M^4)$ be a class of gradient-like flows without heteroclinic intersections such that for any $f^t \in G_{g,1}(M^4)$ the equalities $g = g_{f^t}$, $k_2 = 1$ hold. We are going to describe a class of topological equivalence of any flow $f^t \in G_{g,1}(M^4)$ using the following construction. We prove that the union $W^u_{\Omega^0_{f^t}} \cup W^u_{\Omega^1_{f^t}}$ is a connected graph with g_{f^t} pairwise disjoint cycles. Denote by A_{f^t} a maximal tree of these graph, by R_{f^t} the union of stable invariant of all equilibria of f^t that does not belong to A_{f^t} and set $V_{f^t} = M^4 \setminus (A_{f^t} \cup R_{f^t})$.

Proposition 1. *For any $f^t \in G_{g,1}(M^4)$ there exists a smooth 3-sphere $S^3_{f^t} \subset V_{f^t}$ such that* (see fig. 1)*:*

1. $S^3_{f^t}$ bounds a ball $B^4_{f^t}$ such $A_{f^t} \subset \mathrm{int} B^4_{f^t} \subset (V_{f^t} \cup A_{f^t})$;

2. for any point $x \in V_{f^t}$ the intersection of the trajectory $\mathcal{O}_x$ of x and $S^3_{f^t}$ is transversal.

We set $C^0_{f^t} = S^3_{f^t} \cap \bigcup\limits_{\sigma^1 \in \Omega^1_{f^t}} W^u_{\sigma^1}$, $C^1_{f^t} = S^3_{f^t} \cap W^u_{\sigma^2}$, $C^{2,u}_{f^t} = S^3_{f^t} \cap \bigcup\limits_{\sigma^3 \in \Omega^3_{f^t}} W^u_{\sigma^3}$, $C^{2,s}_{f^t} = S^3_{f^t} \cap \bigcup\limits_{\sigma^1 \in \Omega^1_{f^t}} W^s_{\sigma^1}$ and call the family $K_{f^t} = \{C^0_{f^t}, C^1_{f^t}, C^{2,u}_{f^t}, C^{2,s}_{f^t}\}$ a *Kirby diagram of the flow f^t.*

Corollary 1. *The set $C^0_{f^t}$ is a union of points, $C^1_{f^t}$ is a knot, $C^{2,u}_{f^t}$ and $C^{2,s}_{f^t}$ are disjoint unions of smoothly embedded two-dimensional spheres.*

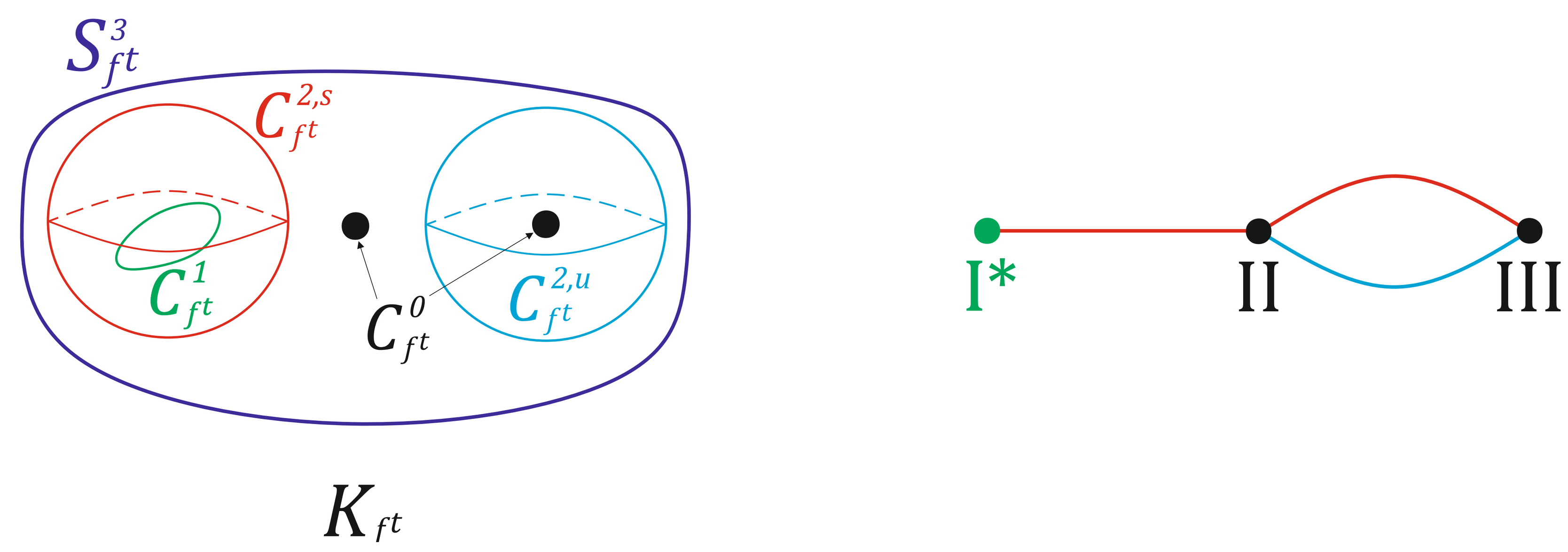

Figure 2: Kirby diagram and bicolor graph of the flow f^t from fig. 1

Lemma 1. *The set $C^1_{f^t}$ is a trivial knot, and the set $\mathrm{cl}\, W^u_p$ is the locally flat two-dimensional sphere.*

We will say that Kirby diagrams $K_{f^t}, K_{f'^t}$ of flows $f^t, f'^t \in G_{g,1}(M^4)$ *are equivalent* if there exists a homeomorphism $h : S^3_{f^t} \to S^3_{f'^t}$ such that:

1. $h(C^i_{f^t}) = C^i_{f'^t}$ for $i \in \{0, 1\}$;

2. $h(C^{2,u}_{f^t}) = C^{2,u}_{f'^t}$, $h(C^{2,s}_{f^t}) = C^{2,s}_{f'^t}$.

Theorem 2. *Let Kirby diagrams $K_{f^t}, K_{f'^t}$ of flows $f^t, f'^t \in G_{g,1}(M^4)$ are equivalent. Then f^t, f'^t are topologically equivalent.*

For any flow $f^t \in G_{g,1}(M^4)$ similarly to [2], [3] we construct a bicolor graph Γ_{f^t} (see fig 2).

Bicolor graphs Γ_{f^t}, $\Gamma_{f'^t}$ of flows $f^t, f'^t \in G_{g,1}(M^4)$ are *isomorphic* if there is an isomorphism $\gamma : \Gamma_{f^t} \to \Gamma_{f'^t}$ preserving colors of edges and a marked vertex.

Theorem 3. *Flows $f^t, f'^t \in G_{g,1}(M^4)$ are topologically equivalent if and only if their bicolor graphs are isomorphic.*

The idea of the proof is to show that an equivalence of bicolor graphs Γ_{f^t}, $\Gamma_{f'^t}$ implies an equivalence of Kirby diagrams K_{f^t}, $K_{f'^t}$.

References

[1] Grines V. Z., Zhuzhoma E. V., Medvedev V. S., *On the structure of the ambient manifold for Morse-Smale systems without heteroclinic intersections*, Trudy Matematicheskogo instituta im. V. A. Steklova RAN, 2017, V. 297, p. 201-210.

[2] Grines V. Z., Gurevich E. Y., *On classification of Morse-Smale flows on projective-like manifolds*, Izvestia RAN, Seriya matematicheskaya, 2022, V. 86 (5), p. 43-72.

[3] Grines V. Z., Gurevich E. Y., *Combinatorial invariant for gradient-like flows on the connected sum $\mathbb{S}^{n-1} \times \mathbb{S}^1$*, Matematicheskiy sbornik, 2023, V. 214, No. 5, p. 97-127.