Manifolds Admitting Polar Flows with Two Saddles of Type (2,2)

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Abstract

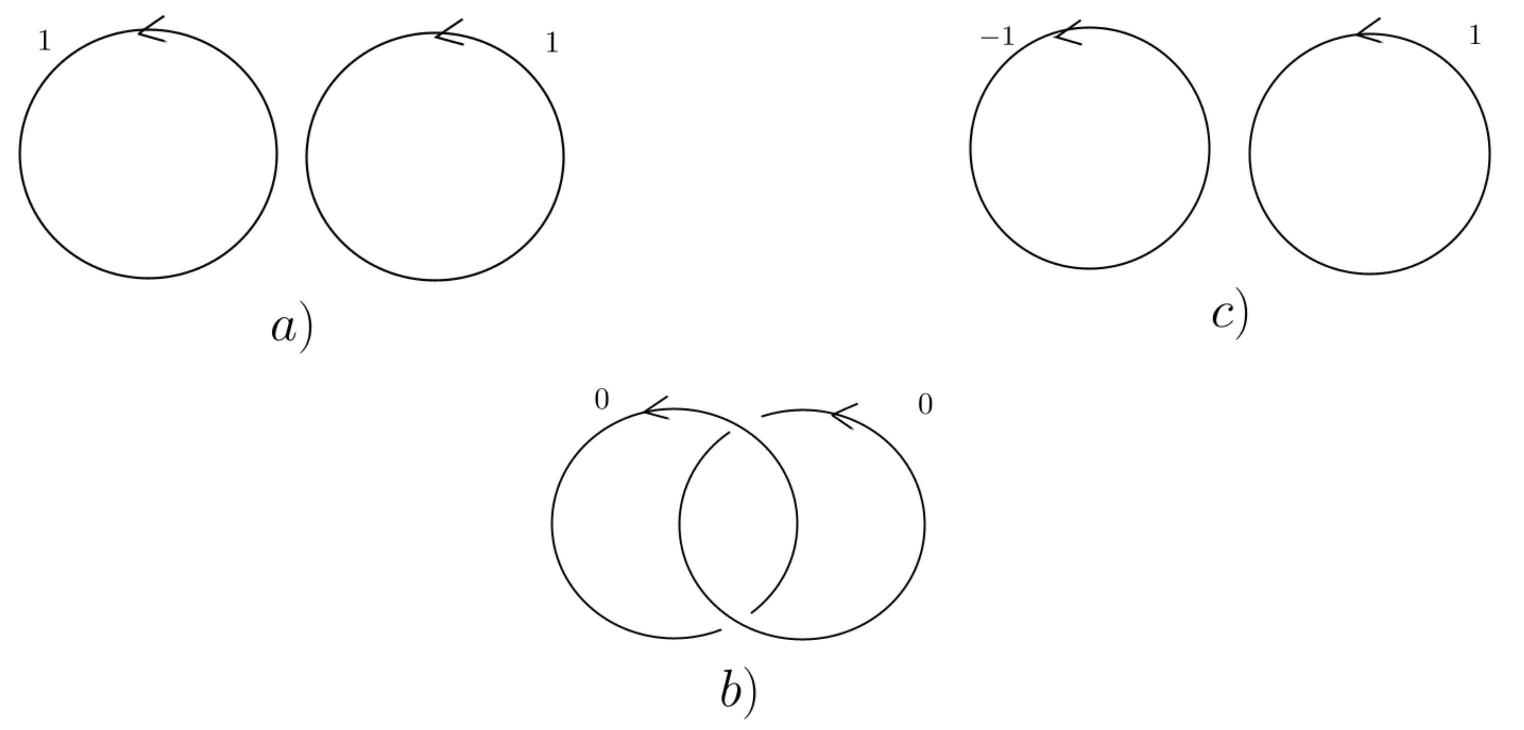
We prove that a closed smooth four-dimensional manifold, admitting a polar flows with two saddles of type $(2,2)$ is homeomorphic to exactly one of three manifolds $\mathbb{C}P^2\sharp\mathbb{C}P^2$, $\mathbb{S}^2\times\mathbb{S}^2$ or $\overline{\mathbb{C}P^2}\sharp\mathbb{C}P^2$.

We recall that a smooth flow $f^t\colon M^n\to M^n$ defined on a closed smooth manifold M^n of dimension n is called a *polar flow* if

1. a non-wandering set Ω_{f^t} of f^t consists exactly of one sink, one source, and a finite number of saddle hyperbolic equilibrium states;
2. invariant manifolds of equilibrium states intersect each other transversely.

The *Morse index* of a hyperbolic equilibrium state p is a number equal to the dimension of its unstable manifold W^u_p.

Let f^t be a polar flow on a manifold M^4 and the set Ω_{f^t} consists of exactly a sink, a source, and two saddles σ_1,σ_2 of Morse index 2. Then M^4 is simply connected, and its homology group $H_2(M^4,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$. According to Freedman's classification of simply connected four-dimensional manifolds (see [1]), the topology of M^4 is determined by a class of equivalence of intersection form, which is an unimodular symmetrical quadratic form $Q: H_2(M^4,\mathbb{Z})\times H_2(M^4,\mathbb{Z})\to\mathbb{Z}$ that put in a correspondence to each elements $x,y\in H_2(M^4,\mathbb{Z})$ their intersection number. For fixed basic of $H_2(M^4,\mathbb{Z})$, the form Q is represented by a symmetric 2×2 matrix A_Q with integer elements. When basis is changed, matrix A_Q is replaced by C^TA_QC, where C is an integer matrix. That is why topology of M^4 is determined, up to orientation preserving homeomorphism, by a classes of congruent (under the integers) unimodular symmetrical matrices. Classification of such matrices was given by Gauss in [2], see also [3]. We find all representatives of congruence classes of determinant matrices by elementary methods and use the result of [4] to find the representatives in the indeterminate case.



Kirby diagrams determing manifolds $\mathbb{C}P^2\sharp\mathbb{C}P^2$, $\mathbb{S}^2\times\mathbb{S}^2$ and $\overline{\mathbb{C}P^2}\sharp\mathbb{C}P^2$

Proposition. *Any binary unimodular integer matrix is congruent to one of the following matrices:*

$$A_1=\begin{pmatrix}1&0\\0&1\end{pmatrix},A_2=\begin{pmatrix}0&1\\1&0\end{pmatrix},A_3=\begin{pmatrix}-1&0\\0&1\end{pmatrix},A_4=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$$

We consrtuct smooth closed manifolds carring polar flows with two saddles and having the intersection forms represented by matricies above in the following way. Let $H^4_k=B^k\times B^{4-k}$, $k\in\{0,\dots,4\}$ and F_k be a vector field on H^4_k given by $\dot{x}=-x,\dot{y}=y$, $x\in B^k,y\in B^{n-k}$. We glue two copies of H^4_2 to H^4_0 by framed links showed on the figures a)-c), above and obtain a manifold with boudnary diffeomorphic to the 3-sphere. Then we attach H^4_4 to this manifold to get the smooth closed manifolds M^4_i. We prove that there exist a basic of $H^4_2(M^4_i)$ such that the matrix of intersection form of M^4_i in this basic coincides with A_1,A_2,A_3, correspondingly. Since $A_4=-A_1$, manifolds, determined by this matricies are gomeomorphic but have the opposite orientation. Due to Freedman's result, the constructed manifolds exhausted the list of all smooth manifolds admitting polar flows with two saddles.

Theorem. *Let M^4 admit a polar flow f^t, non-wandering set of wich consists of exactly a sink, a source, and two saddles σ_1,σ_2 of Morse index 2. Then M^4 is homeomorphic to one of the following manifolds:*

1. a connected sum of two complex projective planes $\mathbb{C}P^2\sharp\mathbb{C}P^2$ with a canonical orientation induced by a complex structure;
2. a direct product $\mathbb{S}^2\times\mathbb{S}^2$ of two copies of two-dimensional spheres.
3. a connected sum $\overline{\mathbb{C}P^2}\sharp\mathbb{C}P^2$ of two copies complex projective planes with opposite orientations.

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