Infinite set of topologically non-equivalent polar flows with two saddles and without heteroclinic intersections

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We recall that a smooth flow $f^t\colon M^n \to M^n$ defined on a closed smooth manifold M^n of dimension n is called a *polar flow* if

1. a non-wandering set Ω_{f^t} of f^t consists exactly of one sink, one source, and a finite number of saddle hyperbolic equilibrium states;
2. invariant manifolds of equilibrium states intersect each other transversely.

The *Morse index* of a hyperbolic equilibrium state p is a number i_p equal to the dimension of its unstable manifold W^u_p. From the Poincaré-Hopf formula[1] it follows that

$$\sum_{p\in\Omega_{f^t}} (-1)^{i_p} = \chi(M^n). \qquad (1)$$

[1] Дж. Милнор, А. Уоллес. Дифференциальная топология. Начальный курс. М.: Мир, 1972. - 279 с. (§6)

A twice-differentiable function $\varphi : M^n \to \mathbb{R}$ is called a *Morse function* if all its critical points are non-degenerate. It follows from Smale's work[2] that for any gradient-like flow f^t on M^n, there exists a *self-indexing energy function* $\varphi : M^n \to [0, n]$ such that:

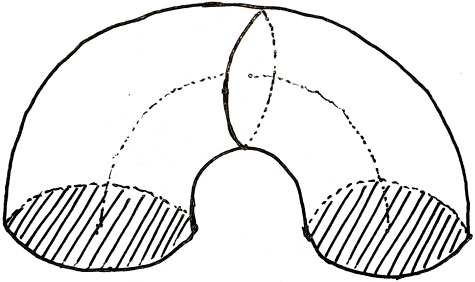
1. The function φ is a Morse function;
2. The set of critical points of φ coincides with the set Ω_{f^t};
3. $\varphi(f^t(x)) < \varphi(x)$ for any point $x \notin \Omega_{f^t}$ and any $t > 0$;
4. $\varphi(p) = i_p$ for any $p \in \Omega_{f^t}$.

[2] S. Smale, "On gradient dynamical systems", Ann. of Math. (2), 74:1 (1961), 199–206.

Definitions

handle

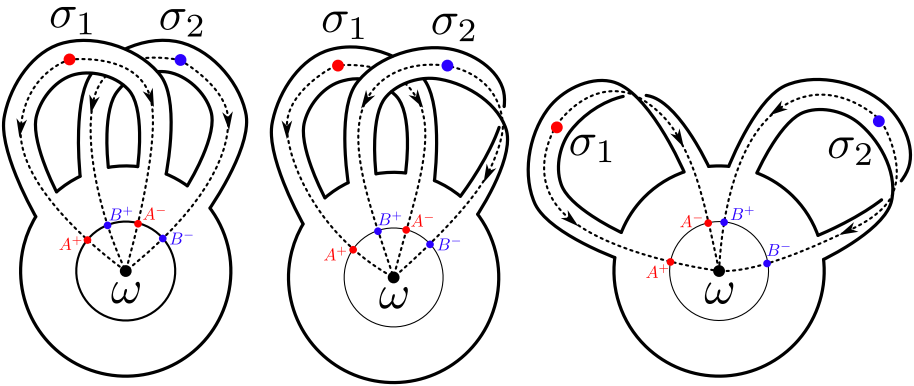
$$H_k^n = \mathbb{B}^k \times \mathbb{B}^{n-k}$$



$$F = \partial\mathbb{B}^k \times \mathbb{B}^{n-k}$$

$n = 2$

Traces of separatrices on a transversal are a complete invariant[3].



[3] G. Fleitas Classification of gradient-like flows on dimensions two and three. Boletim da Sociedade Brasileira de Matemática. 2007, 6, 155-183.

All gradient-like flows on four-dimensional manifolds, whose non-wandering set consists of three equilibrium states, are topologically equivalent[4].

[4] Е. В. Жужома, В. С. Медведев, "Непрерывные потоки Морса–Смейла с тремя состояниями равновесия", Математический сборник, 207:5 (2016), 69-92.

Let $f^t \in P$ and $\varphi : M^4 \to [0,4]$ be an energy function for f^t such that $\varphi(p) = \dim W^u_p$ for any $p \in \Omega_{f^t}$. Put $\Sigma_c = \varphi^{-1}(c)$, $c \in [0,4]$. From the properties of the Morse function, it follows that for any $c \in (0,2) \cup (2,4)$, the set Σ_c is a smoothly embedded three-dimensional sphere, transversal to the trajectories of the flow f^t. Denote by α, ω the source and sink equilibrium states, and by $\Omega^2_{f^t}$ the set of all saddle equilibrium states of the flow f^t. From the condition of transversality of the intersection of invariant manifolds, it follows that $W^s_p \cap W^u_q = \varnothing$ for any $p, q \in \Omega^2_{f^t}, p \neq q$. Then $\mathrm{cl}\, W^u_p = W^u_p \cup \omega, \mathrm{cl}\, W^s_p = W^s_p \cup \alpha$ for any point $p \in \Omega^2_{f^t}$, and the sets $\mathrm{cl}\, W^u_p, \mathrm{cl}\, W^s_p$ are two-dimensional spheres, smoothly embedded in all points, except, possibly, points ω, α respectively.

For $c_1 \in (0,2), c_2 \in (2,4), p \in \Omega^2_{f^t}$ put $l_{p,c_1} = \Sigma_{c_1} \cap W^s_p, l_{p,c_2} = \Sigma_{c_2} \cap W^u_p$. Since the spheres $\Sigma_{c_1}, \Sigma_{c_2}$ intersect with the trajectories of the flow, and, consequently, with the manifolds W^s_p, W^u_q, transversally, the sets l_{p,c_1}, l_{p,c_2} are simple closed smooth curves (knots). Put

$$L_{c_1} = \bigcup_{p \in \Omega^2_{f^t}} l_{p,c_1}, L_{c_2} = \bigcup_{p \in \Omega^2_{f^t}} l_{p,c_2}.$$

Proposition 1

Let the flows $f^t, f'^t \in P$ be topologically equivalent. Then there exists a homeomorphism $h : \Sigma_{c_1} \to \Sigma'_{c_1}$ such that $h(L_{c_1}) = L'_{c_1}$.

To obtain a complete topological invariant, we equip the link L_{c_1} with some additional information. Denote by N_{c_2} the collection of pairwise non-intersecting solid tori, belonging to the sphere Σ_{c_2}, which are closed tubular neighborhoods of the knots forming the link L_{c_2}. Let $\Pi_{p,c_2} \in N_{c_2}$ be the tubular neighborhood of the knot l_{p,c_2} and $\mu_{p,c_2} \subset \partial\Pi_{p,c_2}$ be its meridian (i.e., a simple closed curve, non-homotopic to zero on the boundary $\partial\Pi_{p,c_2}$ of the solid torus Π_{p,c_2}, and bounding a disk $D^2 \subset \Pi_{p,c_2}$). The trajectories of the flow f^t define a diffeomorphism $\eta_{c_1,c_2} : \Sigma_{c_2} \setminus L_{c_2} \to \Sigma_{c_1} \setminus L_{c_1}$, corresponding to any point $x \in \Sigma_{c_2} \setminus L_{c_2}$ with the intersection point of its trajectory $\mathcal{O}_x$ with the sphere Σ_{c_1}. The knot $\tilde{l}_{p,c_1} = \eta_{c_1,c_2}(\mu_{p,c_2})$ will be called the *decoration* of the knot l_{p,c_1}. The collection of decorated knots $\{l_{p,c_1}, \tilde{l}_{p,c_1}\}$ on the sphere Σ_{c_1} will be called the *Kirby diagram of the flow f^t*.

Theorem 1

Flows $f^t, f'^t \in P$ are topologically equivalent if and only if there exists a homeomorphism $h : \Sigma_{c_1} \to \Sigma'_{c_1}$ such that:

1. $h(L_{c_1}) = L'_{c_1}$;
2. $h(\tilde{l}_{p,c_1}) = \tilde{l}'_{p',c_1}$ *for each pair of knots* $l_{p,c_1} \subset L_{c_1}, l'_{p',c_1} = h(l_{p,c_1})$.

Intersection form

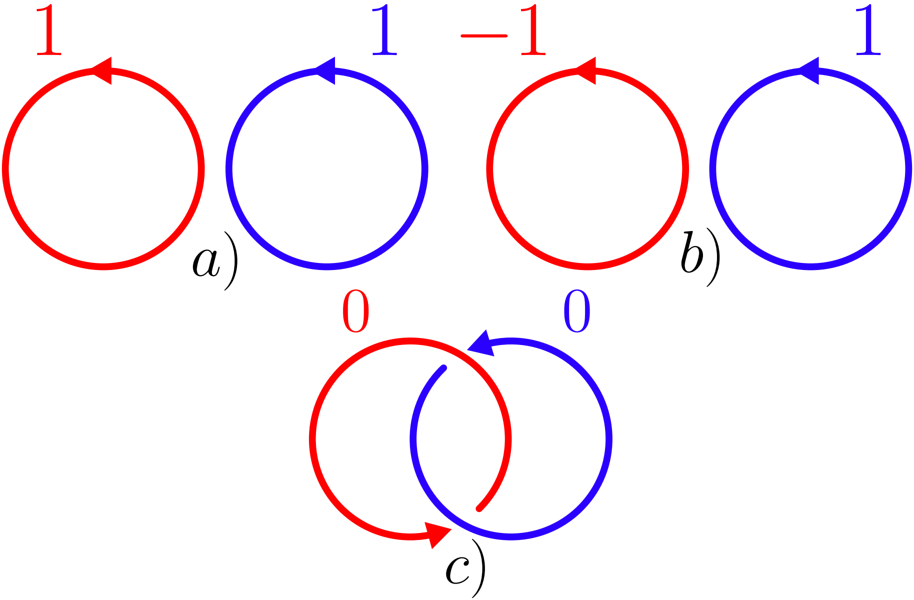
Let f^t be a polar flow on a manifold M^4 and the set Ω_{f^t} consists of exactly a sink, a source, and two saddles σ_1, σ_2 of Morse index 2. Then M^4 is simply connected, and its homology group $H_2(M^4, \mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$. According to Freedman's classification of simply connected four-dimensional manifolds[5], the topology of M^4 is determined by an intersection form, which is an unimodular symmetrical quadratic form $Q : H_2(M^4, \mathbb{Z}) \times H_2(M^4, \mathbb{Z}) \to \mathbb{Z}$ that put in a correspondence to each elements $x, y \in H_2(M^4, \mathbb{Z})$ their intersection number. In some basic of $H_2(M^4)$, the form Q is represented by a symmetric matrices 2×2 with integer elements. That is why topology of M^4 is determined (up to homeomorphsims), by a classes of congruent (under the integers) unimodular symmetrical matrices.

[5] M. Freedman. The Topology of Four-Dimensional Manifolds // J. Diff. Geom. 1982. V. 17. P. 357–453.

Theorem 2

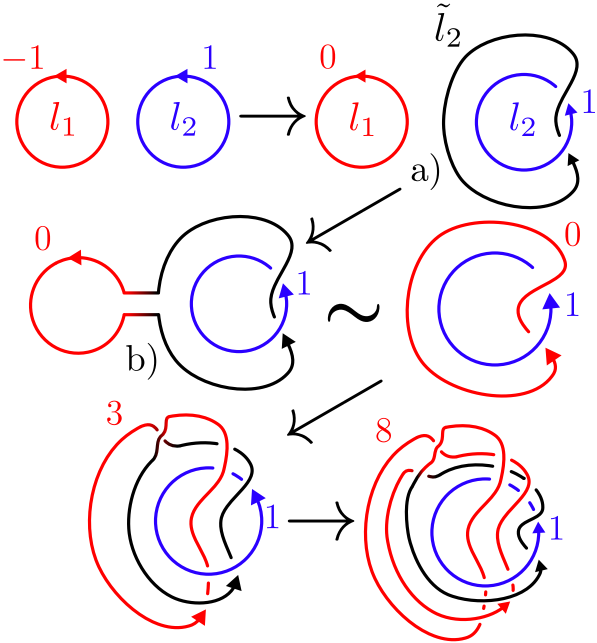
Let M^4 admit a polar flow f^t, non-wandering set of wich consists of exactly a sink, a source, and two saddles σ_1, σ_2 of Morse index 2. Then M^4 is homeomorphic to one of the following manifolds:

1. *a connected sum of two complex projective planes $\mathbb{C}P^2 \# \mathbb{C}P^2$ with a canonical orientation induced by a complex structure;*
2. *a direct product $\mathbb{S}^2 \times \mathbb{S}^2$ of two copies of two-dimensional spheres.*
3. *a connected sum $\overline{\mathbb{C}P^2} \# \mathbb{C}P^2$ of two copies complex projective planes with opposite orientations.*

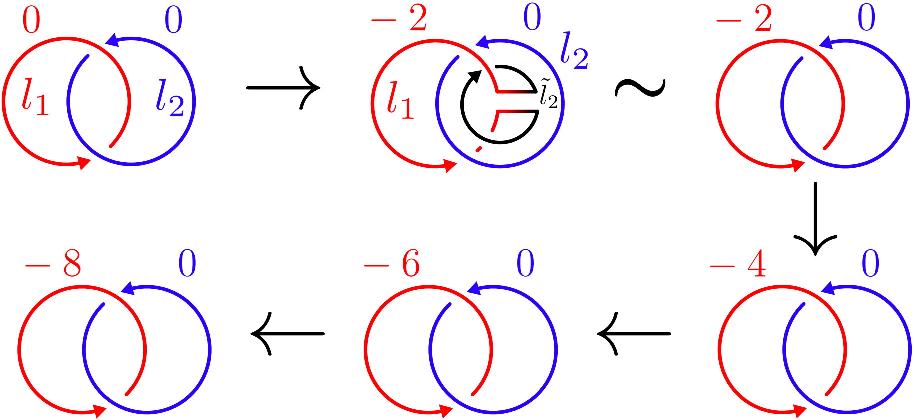

1
1
−1
1
a)
b)
0
0
c)

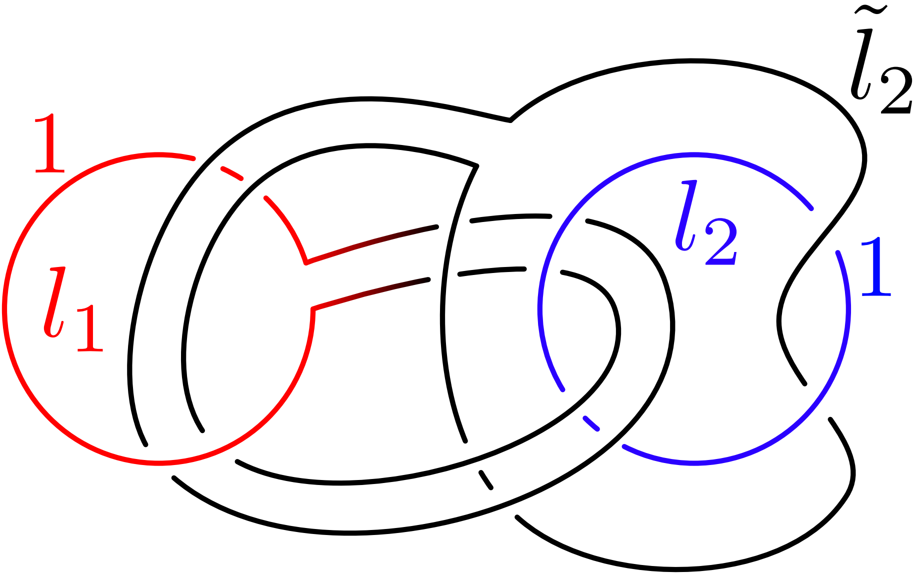
Kirby move

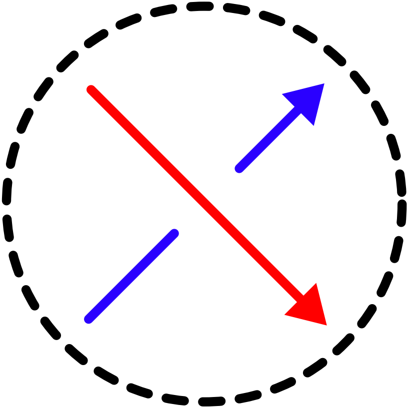
Let $L \subset S^3$ be a framed link, where the knots $\lambda_i, \lambda_j \subset L$ have framings $\widetilde{\lambda}_i, \widetilde{\lambda}_j$ with framing coefficients n_i, n_j, respectively. Suppose $e : [0,1] \times [0,1] \to S^3$ is a smooth embedding such that the intersection of the set $R = e([0,1] \times [0,1])$ with L consists of two segments $l_0 = e(\{0\} \times [0,1]) \subset \lambda_i$ and $l_1 = e(\{1\} \times [0,1]) \subset \widetilde{\lambda}_j$. The knot $\lambda_i' = \lambda_i \cup \widetilde{\lambda}_j \cup \partial R \setminus \mathrm{int}(l_0 \cup l_1)$ is called the *band sum* of knots λ_i and $\widetilde{\lambda}_j$. Let $n_i' = n_i + n_j \pm 2\mathrm{lk}(\lambda_i, \lambda_j)$, where the sign "$+$"is taken if the orientation of the segments l_0, l_1 induced by the orientation of the boundary of the ribbon $[0,1] \times [0,1]$ is opposite to the orientation of both knots $\lambda_i, \widetilde{\lambda}_j$, and the sign "$-$"is taken otherwise. The replacement in the framed link L of the pair $(\lambda_i, \widetilde{\lambda}_i)$ by the pair $(\lambda_i', \widetilde{\lambda}_i')$ with the framing coefficient n_i' is called a *Kirby move of the second type*. The Kirby move of the second type corresponds to *dragging the handle* A_i attached along the knot λ_i along the handle A_j attached along the knot λ_j. During this operation, the bottom of the handle A_i is smoothly shifted by an isotopy $h_t : \partial(B^4 \cup A_j) \to \partial(B^4 \cup A_j)$ along the band R.



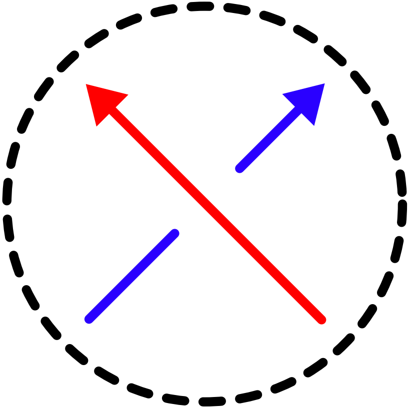
-1
l_1
1
l_2
0
l_1
$\tilde{l}_2$
l_2
1
a)
0
1
$\sim$
0
1
b)
3
1
8
1


0
0
l_1
l_2
-2
0
l_1
l_2
$\tilde{l}_2$
-2
0
-8
0
-6
0
-4
0


$\tilde{l}_2$
1
l_1
l_2
1



$(+)$



$(-)$

Thank you for your attention!